Brauer containers

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Motivation

Picard group in algebraic geometry

- In algebraic geometry we study **algebraic varieties**, which are the set of solutions of a system of polynomial equations
- A variety V is equipped with a **topology**—a collection of subsets termed as open subsets—and a **structure sheaf** $\mathcal{O}_V$ of rings, associating a ring to each open subset compatibly with restriction of subsets
- A **module** over a ring is a set that the ring acts on linearly, like a vector space over a field
- We refer to **a sheaf of modules over** $\mathcal{O}_V$ as a V-module
- From the tensor product $\otimes$ of modules, we can define a tensor product $\otimes$ of V-modules
- The **Picard group** of V is the group of isomorphism classes of invertible V-modules or **line bundles** (operation is $\otimes$, identity element is $\mathcal{O}_V$)

Motivation

Jacobian of a complex curve

- The Picard group can be given the structure of an algebraic variety $\mathrm{Pic}(V)$
- Varieties like $\mathrm{Pic}(V)$ that are also abelian groups are called **abelian varieties**
- Over $\mathbb{C}$, any complex abelian variety is a **complex torus** $\mathbb{C}^d/\Gamma$ that admits an algebraic structure
- If C is a curve over the complex numbers, then the connected component of the identity in $\mathrm{Pic}(C)$ is the **Jacobian variety**
- In addition to this algebraic definition, we can give an analytic construction of the Jacobian variety

Motivation

Analytic construction of the Jacobian

- Given a complex curve C we have a short exact sequence of sheaves (called the **exponential sequence**)

$$\mathbb{Z} \hookrightarrow \mathcal{O}_C \overset{\exp}{\twoheadrightarrow} \mathcal{O}_C^{\times}$$

- This gives rise to a long exact sequence of sheaf cohomology groups

$$0 \to \mathrm{H}^0(C, \mathbb{Z}) \to \mathrm{H}^0(C, \mathcal{O}_C) \to \mathrm{H}^0(C, \mathcal{O}_C^{\times}) \to \mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathcal{O}_C) \to \dots$$

- The cokernel of $\mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathcal{O}_C)$ is a **complex torus** $\mathbb{C}^g/\Gamma$ (Γ is a discrete subgroup of $\mathbb{C}^g$)
- This torus gives an **analytic construction** of the Jacobian variety
- This torus is algebraic, which makes it an **abelian variety**

Motivation

Higher dimensional complex varieties

- The Jacobian of a complex curve can be generalized to give an algebraic complex torus structure to the cokernel of $\mathrm{H}^1(X, \mathbb{Z}) \to \mathrm{H}^1(X, \mathcal{O}_X)$ for higher-dimensional X
- If X is smooth and projective, this construction is a complex torus because the image of $\mathrm{H}^1(X, \mathbb{Z})$ in $\mathrm{H}^1(X, \mathcal{O}_X)$ is **discrete**
- Is the cokernel of $\mathrm{H}^m(X, \mathbb{Z}) \to \mathrm{H}^m(X, \mathcal{O}_X)$ also a complex torus for $m > 1$?
- If $m > 1$,
  1. the image of $\mathrm{H}^m(X, \mathbb{Z})$ in $\mathrm{H}^m(X, \mathcal{O}_X)$ is **not necessarily discrete**, so the cokernel of $\mathrm{H}^m(X, \mathbb{Z}) \to \mathrm{H}^m(X, \mathcal{O}_X)$ is **not** necessarily a complex torus;
  2. even if it is a complex torus, it is **not** necessarily algebraic

Motivation

Degree 2: the Brauer container

- If the rank of the group $\mathrm{Pic}(X)$ is **maximal**, then the cokernel of $\mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{O}_X)$ is a complex torus that we denote $\mathfrak{C}(X)$

- The torsion subgroup of $\mathfrak{C}(X)$ is isomorphic to the Brauer group $\mathrm{Br}(X)$, so we call $\mathfrak{C}(X)$ the **Brauer container**

- $\mathfrak{C}(X)$ has been studied by Beauville, Shioda & Mitani, among others

- When X is an abelian variety, $\mathfrak{C}(X)$ has been computed up to **isogeny** (a surjective morphism with finite kernel)

- Furthermore, in this case $\mathfrak{C}(X)$ is algebraic (an abelian variety)

Motivation

Products of elliptic curves

- An **elliptic curve** is a 1-dimensional abelian variety, which is often described as the solution set of an equation $y^2 = x^3 + ax + b$
- We say the elliptic curve has **complex multiplication (CM)** if it has more endomorphisms than just multiplication by $\mathbb{Z}$
- (Schoen) A complex abelian variety X of maximal Picard rank is (not uniquely) isomorphic to a **product of pairwise isogenous CM elliptic curves**

$$X \cong E_1 \times E_2 \times \ldots \times E_n$$

- In the $n = 2$ case, Shioda and Mitani show that $\mathfrak{C}(E_1 \times E_2)$ is an elliptic curve that is isogenous to both E_1 and E_2

Goal

- (Schoen) A complex abelian variety X of maximal Picard rank is (not uniquely) isomorphic to a **product of pairwise isogenous CM elliptic curves**

$$X \cong E_1 \times E_2 \times \ldots \times E_n$$

- In the $n = 2$ case, Shioda and Mitani show that $\mathfrak{C}(E_1 \times E_2)$ is an elliptic curve that is isogenous to both E_1 and E_2

- We will use this concrete description of X to compute $\mathfrak{C}(X)$ up to **isomorphism**, via number theory

- We will show that for such X, the cokernel of $\mathrm{H}^m(X, \mathbb{Z}) \to \mathrm{H}^m(X, \mathcal{O}_X)$ for **all** $m \leq n$ is also a complex abelian variety of maximal Picard number

Roadmap

- **Number theory**
  - Elliptic curves over $\mathbb{C}$ and lattices
  - Complex multiplication and the ideal class group
- Computing containers
- Fields of definition
  - Issues with extending to fields other than $\mathbb{C}$

Number theory

Lattices in the complex numbers

- A **lattice** $\Gamma \subseteq \mathbb{C}$ is the set of $\mathbb{Z}$-linear combinations of some $v, w \in \mathbb{C}$ with $v/w \notin \mathbb{R}$:

$$\Gamma = \mathbb{Z}v + \mathbb{Z}w = \langle v, w \rangle$$

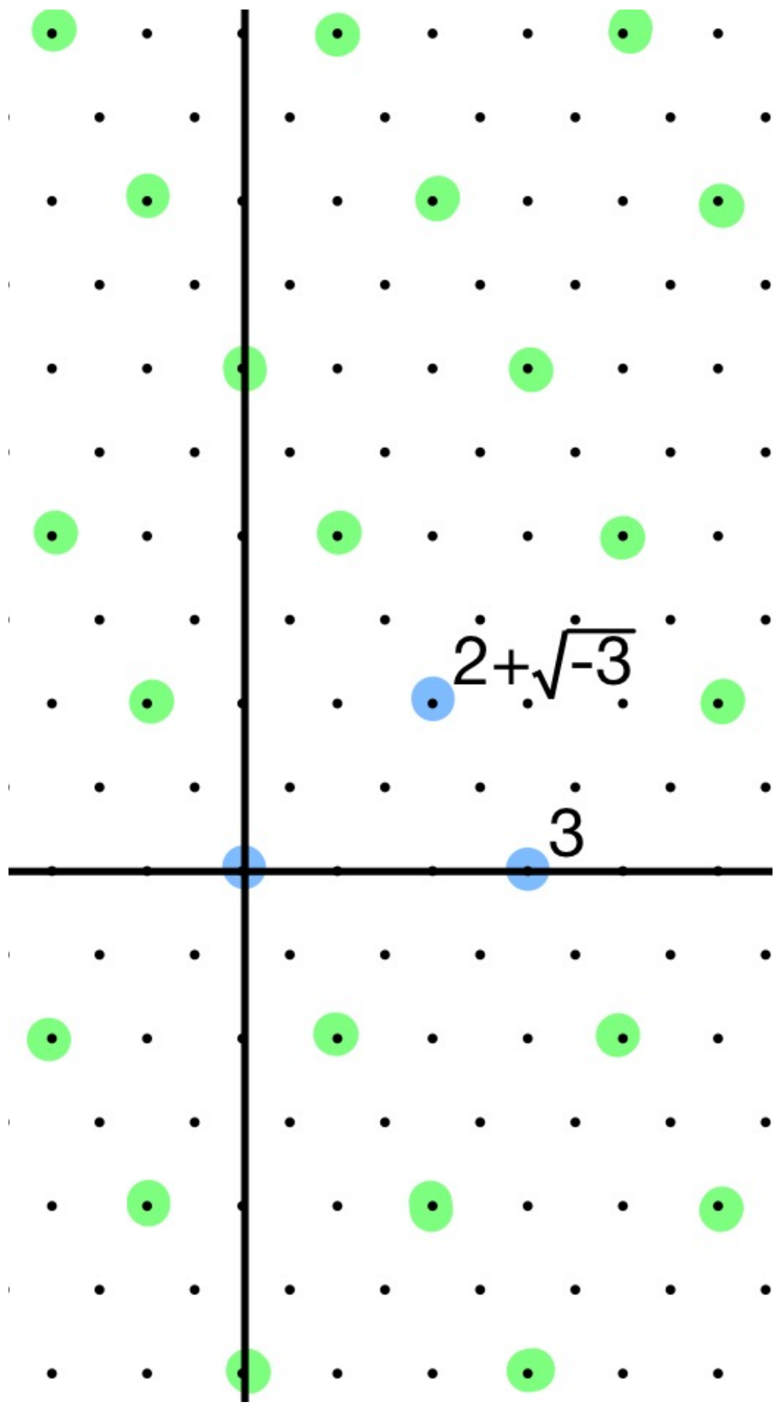
- Any complex elliptic curve E is isomorphic as a Lie group to a torus $\mathbb{C}/\Gamma$ for a lattice $\Gamma \subseteq \mathbb{C}$

- An isomorphism of elliptic curves $E \cong \mathbb{C}/\Gamma, E' \cong \mathbb{C}/\Gamma'$ corresponds to $\alpha \in \mathbb{C}$ with $\alpha\Gamma = \Gamma'$

- **Isomorphism** is **homothety** (rotation, scaling) of lattices

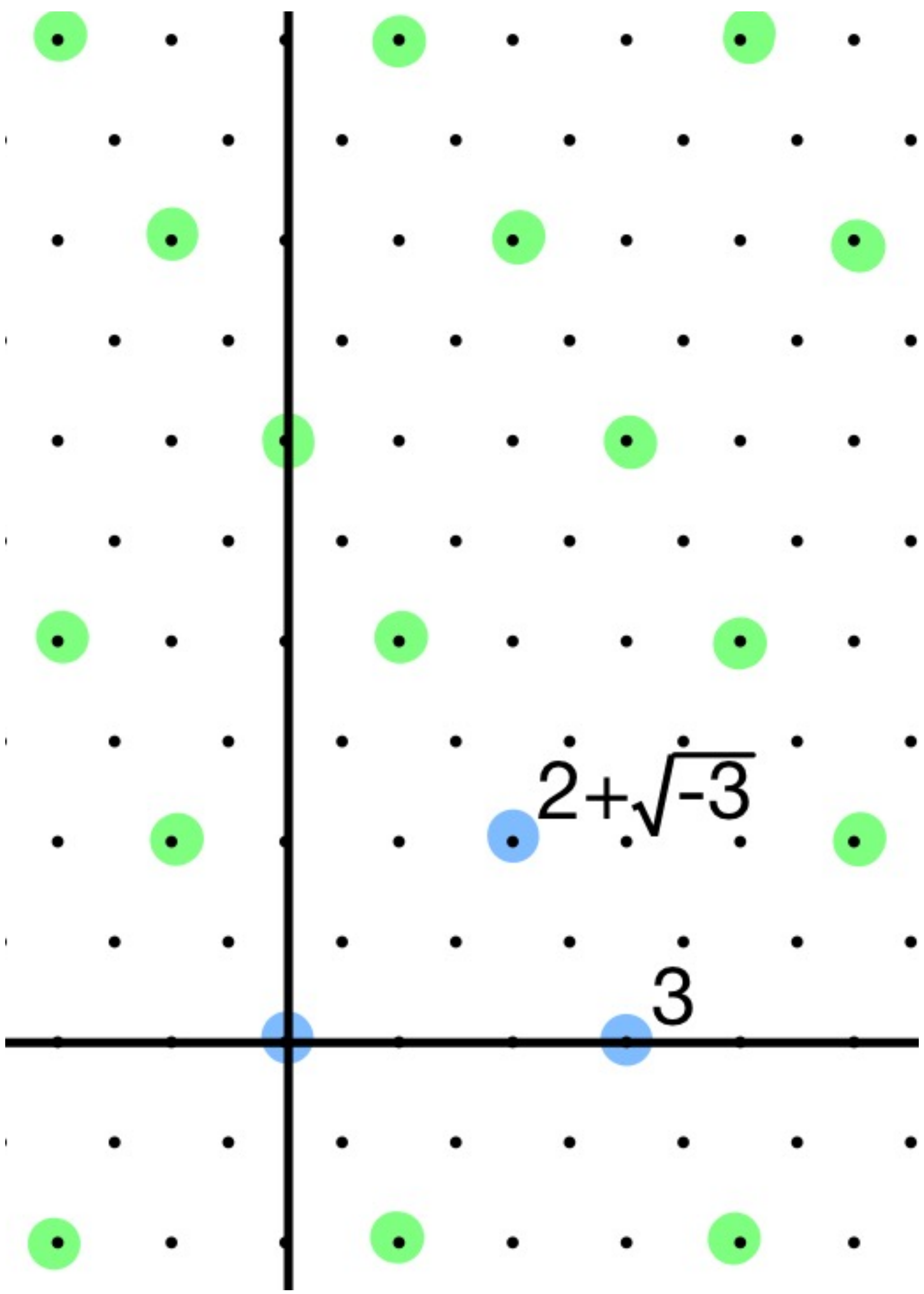
- An isogeny of elliptic curves corresponds to $\alpha \in \mathbb{C}$ with $\alpha\Gamma \subseteq \Gamma'$

- Two lattices are **isogenous** if one is **homothetic to a sublattice** of the other



Number theory

Complex multiplication



- The **endomorphism ring** of a lattice is the ring of isogenies to itself

$$\mathrm{End}(\Gamma) = \{\alpha \in \mathbb{C} : \alpha\Gamma \subseteq \Gamma\}$$

- If $\mathrm{End}(\Gamma)$ is larger than $\mathbb{Z}$ we say that the lattice/elliptic curve has **complex multiplication (CM)** by $\mathrm{End}(\Gamma)$

- For $\Lambda = \langle 3, 2+\sqrt{-3}\rangle$ we have

$$\mathrm{End}(\Lambda) = \mathbb{Z}[3\sqrt{-3}]$$

- Λ is homothetic to $3\Lambda = \langle 9, 6+3\sqrt{-3}\rangle \subseteq \mathbb{Z}[3\sqrt{-3}]$, which is an ideal of $\mathbb{Z}[3\sqrt{-3}]$

Number theory

CM elliptic curves and the class group

- If $E \cong \mathbb{C}/\Gamma$ has CM, then Γ is homothetic to an ideal of $\mathrm{End}(\Gamma) = \mathrm{End}(E)$
- Think of Γ as a (fractional) ideal of $R = \mathrm{End}(\Gamma)$, and homothety as multiplication by an element of R—or by a principal ideal
- The **ideal class group** $\mathrm{Cl}(R)$ is the group of (invertible) fractional ideals of R modulo principal ideals (R is the identity element)
- Homothety classes of lattices with CM by R correspond to elements of $\mathrm{Cl}(R)$
- We use $[E] = [\Gamma] \in \mathrm{Cl}(R)$ for the group element corresponding to the homothety/isomorphism class of $E \cong \mathbb{C}/\Gamma$

Roadmap

- Number theory
- **Computing containers**
  - Lattices and the Brauer container
  - Self-product example $E \times E$
  - Brauer container of an abelian surface
  - Brauer containers of abelian n-folds
  - The n-container of an abelian n-fold
  - Computing the m-container
- Fields of definition
  - Issues with extending to fields other than $\mathbb{C}$

Computing containers

Isogeny of CM lattices

- Recall: a complex abelian variety X of maximal Picard number is isomorphic to a **product of pairwise isogenous CM elliptic curves** (Schoen)

$$X \cong E_1 \times E_2 \times \ldots \times E_n$$

- What does this mean in terms of lattices?
- If Γ is a CM lattice with $R = \mathrm{End}(\Gamma)$, then the **endomorphism algebra** $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$
- In fact, R is a subring of the endomorphism algebra $K = R \otimes_{\mathbb{Z}} \mathbb{Q}$
- Two CM elliptic curves E_1, E_2 are isogenous **if and only if** the endomorphism algebras are the same

Computing containers

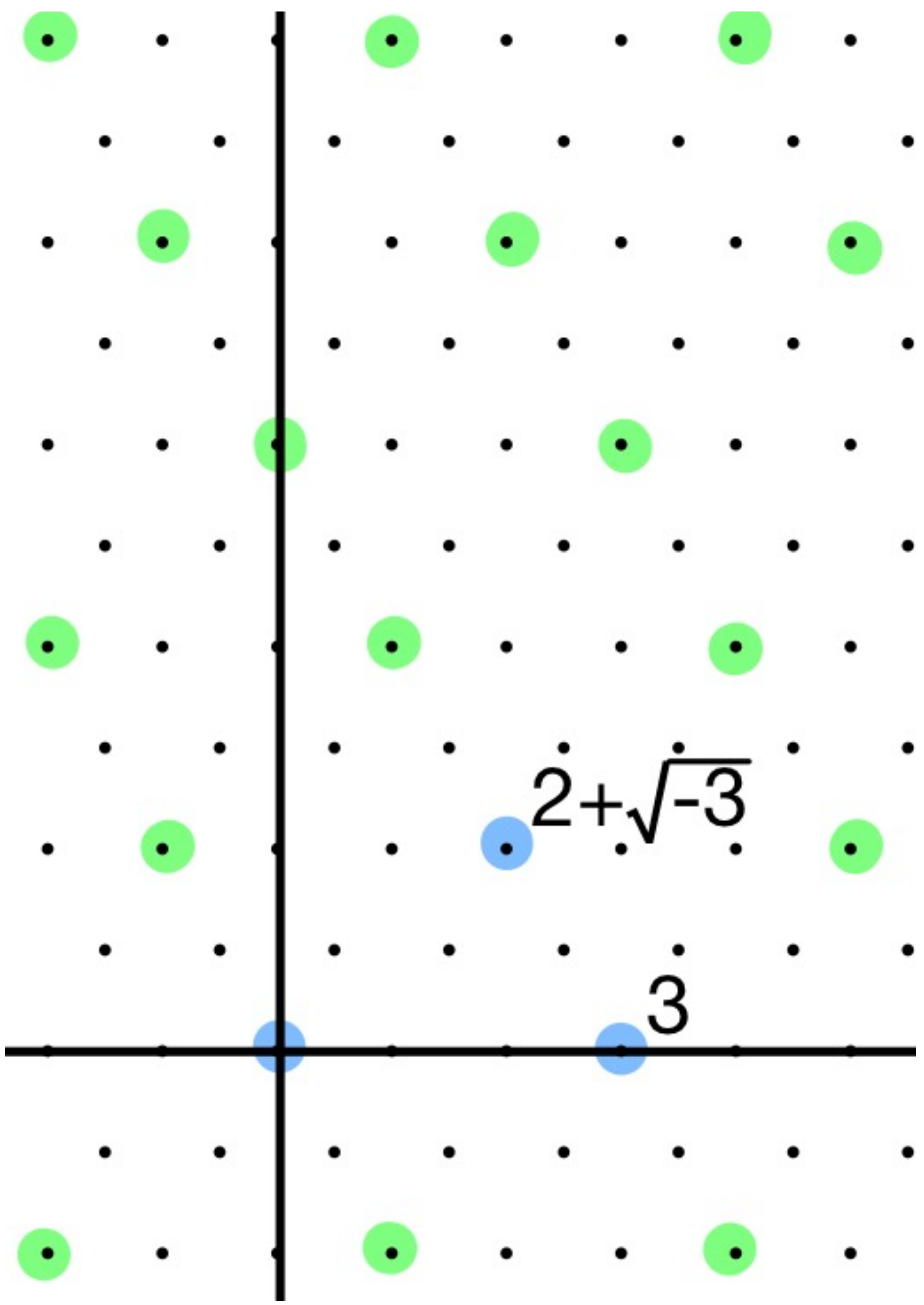
Lattices and the Brauer container

- Shioda and Mitani found that $\mathfrak{C}(E_1 \times E_2)$ is a CM elliptic curve **isogenous** to both E_1, E_2
- To compute $\mathfrak{C}(E_1 \times E_2)$ up to **isomorphism**, we'll find $R = \operatorname{End}(\mathfrak{C}(E_1 \times E_2))$ and an element in $\operatorname{Cl}(R)$ for the isomorphism class $[\mathfrak{C}(E_1 \times E_2)]$
- It's known that if $E_i \cong \mathbb{C}/\langle v_i, w_i \rangle$ and $v_1v_2, v_1w_2, w_1v_2, w_1w_2 \in \mathbb{C}$ span a lattice Γ then $\mathfrak{C}(E_1 \times E_2) \cong \mathbb{C}/\Gamma$

Computing containers

Self-product example

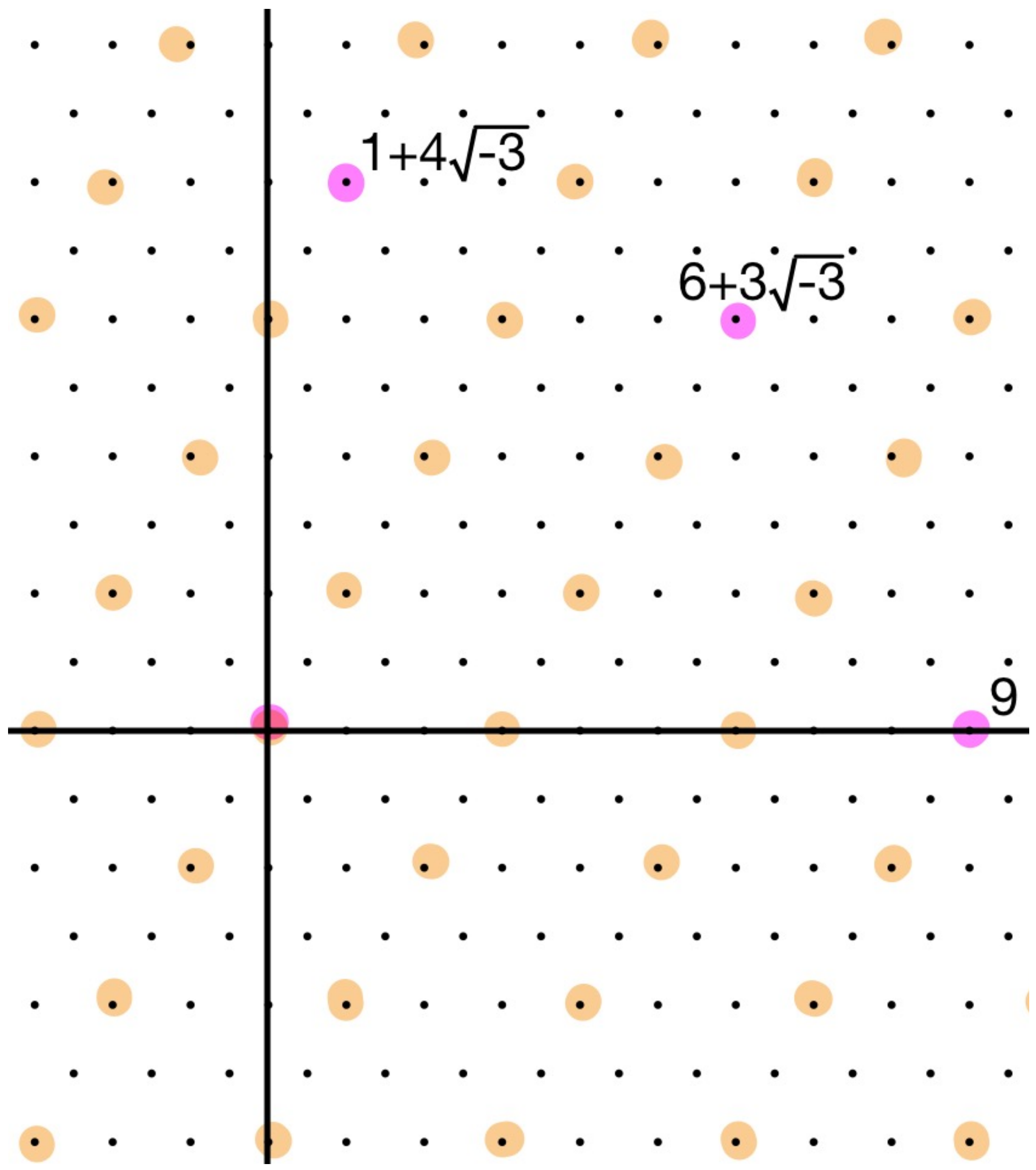
- Consider the elliptic curve $E = \mathbb{C}/\Lambda$ corresponding to the lattice $\Lambda = \langle 3, 2 + \sqrt{-3} \rangle$
- $\mathfrak{C}(E \times E)$ is isomorphic to an elliptic curve $E' \cong \mathbb{C}/\Lambda'$
- Λ' is spanned by
  - $3^2 = 9$
  - $3 \cdot (2 + \sqrt{-3}) = 6 + 3\sqrt{-3}$
  - $(2 + \sqrt{-3})^2 = 1 + 4\sqrt{-3}$



Computing containers

Self-product example

- $\{9, 6+3\sqrt{-3}, 1+4\sqrt{-3}\}$ span the lattice $\Lambda' = \langle 3, 1+\sqrt{-3}\rangle$
- Hence $\mathfrak{C}(E \times E)$ is isomorphic to $E' = \mathbb{C}/\Lambda'$
- Λ, Λ' are **not homothetic**, so $E \not\cong E' \cong \mathfrak{C}(E \times E)$
- However, they are isogenous; both have endomorphism ring $\mathbb{Z}[3\sqrt{-3}]$



Computing containers

Brauer container of an abelian surface

- The easiest case to compute the isomorphism class $[\mathfrak{C}(E_1 \times E_2)]$ is when $\mathrm{End}(E_1) = \mathrm{End}(E_2) = R$
- In that case, $\mathfrak{C}(E_1 \times E_2)$ **also has CM by** R and

$$[\mathfrak{C}(E_1 \times E_2)] = [E_1][E_2] \in \mathrm{Cl}(R)$$

- More generally, if $\mathrm{End}(E_i) = R_i$, then we can show $\mathfrak{C}(E_1 \times E_2)$ **has CM by** $R_0 := R_1 + R_2 \subseteq K = R_i \otimes_{\mathbb{Z}} \mathbb{Q}$
- Letting $E_i \cong \mathbb{C}/\Gamma_i$ we find that $\Gamma_i \otimes_{R_i} R_0$ is a lattice with CM by R_0 and

$$[\mathfrak{C}(E_1 \times E_2)] = [\Gamma_1 \otimes_{R_1} R_0][\Gamma_2 \otimes_{R_2} R_0] \in \mathrm{Cl}(R_0)$$

Computing containers

Brauer containers of abelian *n*-folds

- Recall: a complex abelian variety X of maximal Picard number is isomorphic to a **product of pairwise isogenous CM elliptic curves** (Schoen)

$$X \cong E_1 \times E_2 \times \ldots \times E_n$$

- Computing directly with the map $\mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{O}_X)$ we find that

$$\mathfrak{C}(X) \cong \prod_{i<j} \mathfrak{C}(E_i \times E_j)$$

- This means $\mathfrak{C}(X)$ is a product of CM elliptic curves, hence an abelian variety of dimension $\binom{n}{2}$

- In fact, the $\mathfrak{C}(E_i \times E_j)$ are also pairwise isogenous, so $\mathfrak{C}(X)$ is **also a complex abelian variety of maximal Picard rank!**

Computing containers

The *n*-container of an abelian *n*-fold

- For $X \cong E_1 \times \ldots \times E_n$ an abelian variety of maximal Picard number, the cokernel of $\mathrm{H}^m(X, \mathbb{Z}) \to \mathrm{H}^m(X, \mathcal{O}_X)$ is **also a complex torus**

- We call this torus $\mathfrak{C}_m(X)$ or the ***m*-container**

  - The Brauer container $\mathfrak{C}(X)$ is the 2-container

- If $n = m$, then we find similarly that $\mathfrak{C}_n(E_1 \times \ldots \times E_n)$ is an elliptic curve with CM by R, where R is

$$R_1 + \ldots + R_n \subseteq K = R_i \otimes_{\mathbb{Z}} \mathbb{Q}, \text{ for } R_i = \mathrm{End}(E_i)$$

- We can compute $\mathfrak{C}_n(X)$ up to isomorphism as

$$[\mathfrak{C}_n(X)] = \prod_{i=1}^{n} [\Gamma_i \otimes_{R_i} R] \in \mathrm{Cl}(R), \text{ for } \Gamma_i \text{ such that } E_i \cong \mathbb{C}/\Gamma_i$$

Computing containers

Computing the *m*-container

- For $m < n$, we find that the ***m*-container** of X is

$$\mathfrak{C}_m(X) \cong \prod_{i_1 < i_2 < \ldots < i_m} \mathfrak{C}_m\left(E_{i_1} \times \ldots \times E_{i_m}\right)$$

- Each $\mathfrak{C}_m(E_{i_1} \times \ldots \times E_{i_m})$ is a **CM elliptic curve isogenous to the E_i**

- $\mathfrak{C}_m(X)$ is a product of pairwise isogenous CM elliptic curves

- Therefore, for any X which is a complex abelian variety of maximal Picard rank, the *m*-containers $\mathfrak{C}_m(X)$ for $m \leq \dim(X)$ are also **complex abelian varieties of maximal Picard rank!**

- In particular, $\mathfrak{C}_{n-1}(X)$ is a complex abelian variety of the same dimension n

Roadmap

- Number theory
- Computing containers
- **Fields of definition**
    - Issues with extending to fields other than $\mathbb{C}$
    - Return to self-product example
        - $\mathfrak{C}(E \times E)$ cannot be defined over the minimal field of definition of E
    - Elliptic curves and the ring class field
        - Give a condition on the fields of definition of two elliptic curves with class field theory
    - Finding other interesting examples

Field of definition

Issues extending to other fields

- Question: is there an analogue of the Brauer container for varieties of maximal Picard number over **fields other than** $\mathbb{C}$?
- For example, if we have an abelian surface $\mathscr{A}$ over a number field L, is there **an elliptic curve** $E_{\mathscr{A}}$ **over** L with some nice relationship to the Brauer group of $\mathscr{A}$?
- We would expect that $E_{\mathscr{A}} \otimes \mathbb{C} \cong \mathfrak{C}(\mathscr{A}_{\mathbb{C}})$, but sometimes **no such** $E_{\mathscr{A}}$ **over** L **exists!**
- This means though $\mathfrak{C}(\mathscr{A}_{\mathbb{C}})$ is algebraic (it is an abelian variety), the construction of $\mathfrak{C}(\mathscr{A}_{\mathbb{C}})$ is **not-so-algebraic**

Field of definition

j-invariants of elliptic curves

- For a complex elliptic curve E defined by an equation $y^2 = x^3 + ax + b$, the j-**invariant** is

$$j(E) = 1728\frac{4a^3}{4a^3 + 27b^2}$$

- A complex elliptic curve is determined up to isomorphism by its j-invariant, and E is always isomorphic over $\mathbb{C}$ to the elliptic curve defined by

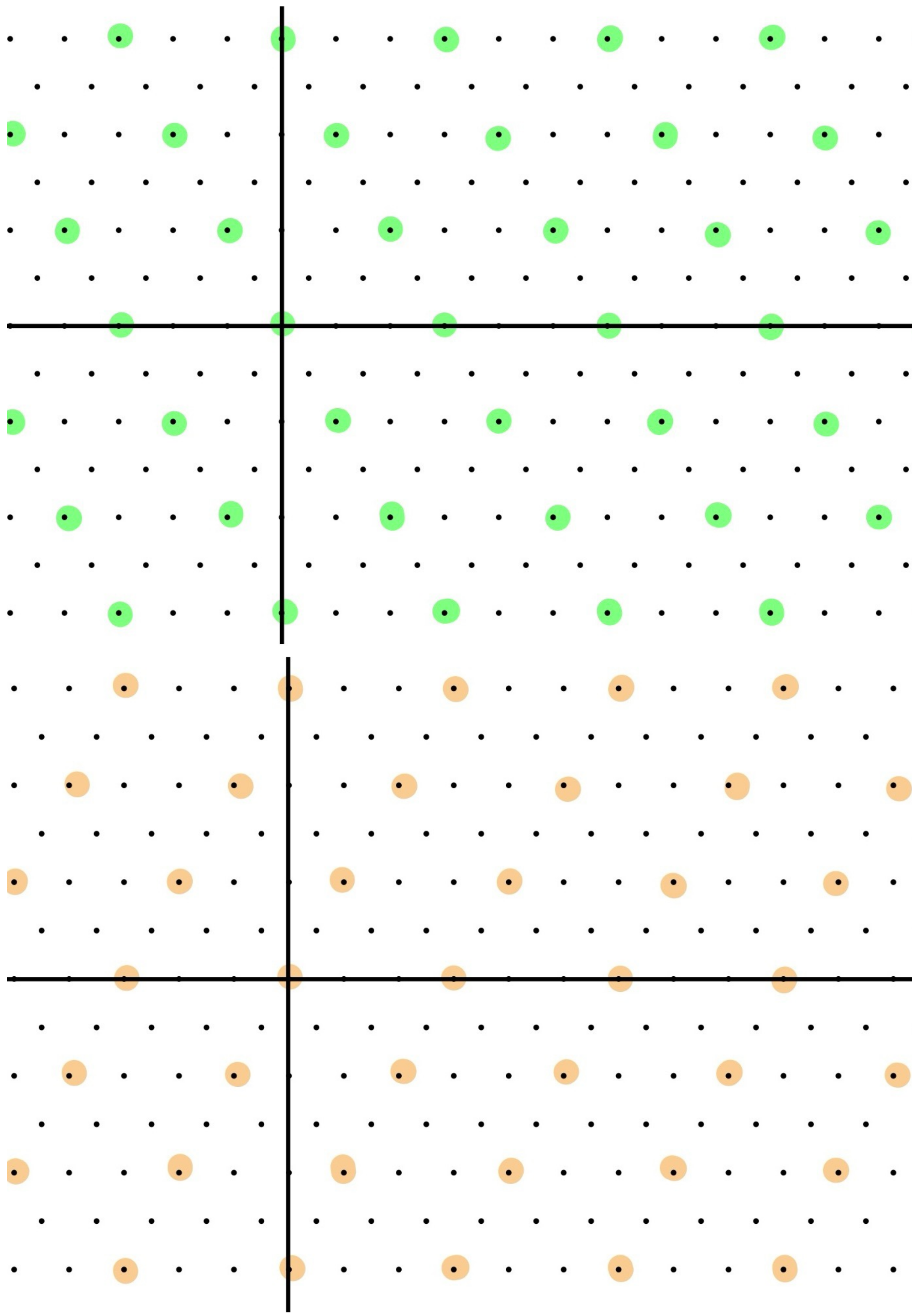
$$y^2 = x^3 + \frac{3j(E)}{1728 - j(E)}x + \frac{2j(E)}{1728 - j(E)}$$

- $\mathbb{Q}(j(E))$ is the "smallest" number field over which E can be defined

- We will give an example of an abelian surface $\mathscr{A}$ over a number field L so that $j(\mathfrak{C}(\mathscr{A}_{\mathbb{C}})) \notin L$

Field of definition

Return to self-product example



- Recall the elliptic curve $E = \mathbb{C}/\Lambda$ corresponding to the lattice $\Lambda = \langle 3, 2 + \sqrt{-3} \rangle$
- $\mathfrak{C}(E \times E)$ is isomorphic to an elliptic curve $E' = \mathbb{C}/\Lambda'$ with $\Lambda' = \langle 3, 1 + \sqrt{-3} \rangle$
- Λ, Λ' are **not homothetic** so $E \not\cong E'$
- By computation, $\mathbb{Q}(j(E)) = \mathbb{Q}(\zeta_3 \sqrt[3]{2})$ and $\mathbb{Q}(j(E')) = \mathbb{Q}(\zeta_3^2 \sqrt[3]{2})$
- Since $j(\mathfrak{C}(E \times E)) \notin \mathbb{Q}(j(E))$, we can define $E \times E$ over this field $\mathbb{Q}(j(E)) = \mathbb{Q}(\zeta_3 \sqrt[3]{2})$, but we **can't define its Brauer container!**

Field of definition

Self-product example and the class group

- We will find other examples of an elliptic curve E such that E can be defined over a number field L but $\mathfrak{C}(E \times E)$ cannot, using the class group of $\mathrm{End}(E)$

- Let's look at $\mathrm{Cl}(\mathrm{End}(E))$ for our previous example $E = \mathbb{C}/\Lambda$ where $\Lambda = \langle 3, 2 + \sqrt{-3} \rangle$

- In this case, $\mathrm{End}(E) = \mathbb{Z}[3\sqrt{-3}]$ and $\mathrm{Cl}(\mathbb{Z}[3\sqrt{-3}]) \cong \mathbb{Z}/3\mathbb{Z}$

- Note that $\mathbb{Z}[3\sqrt{-3}] = \langle 1, 3\sqrt{-3} \rangle$ is the lattice corresponding to the identity element and $[E], [E]^2 = [\mathfrak{C}(E \times E)] = [E']$ are the non-identity elements

- $\mathbb{Q}(j(\mathbb{C}/\langle 1, 3\sqrt{-3} \rangle)) = \mathbb{Q}(\sqrt[3]{2})$, which is distinct from both $\mathbb{Q}(j(E)), \mathbb{Q}(j(E'))$

- All elements of the class group have **different minimal fields of definition**

Field of definition

Elliptic curves and the ring class field

- For two CM elliptic curves E_1, E_2, we want to describe when $j(E_2) \in \mathbb{Q}(j(E_1))$
- For **any** elliptic curve E with CM by $R \subseteq R \otimes_{\mathbb{Z}} \mathbb{Q} = K$ the extension $K(j(E))$ is equal to the **ring class field** L_R
- L_R is a degree $n = |\mathrm{Cl}(R)|$ extension of K
- We use class field theory to prove a **field of definition condition**: if E_1, E_2 both have CM by R then either
  - (i) $[E_1]^2 = [E_2]^2 \in \mathrm{Cl}(R)$ and $\mathbb{Q}(j(E_1)) = \mathbb{Q}(j(E_2))$ is a degree n extension of $\mathbb{Q}$, or
  - (ii) $j(E_2) \notin \mathbb{Q}(j(E_1))$ and $\mathbb{Q}(j(E_1), j(E_2)) = L_R$

Field of definition

Other interesting examples

- We can find many cases of an elliptic curve E such that E can be defined over a number field L but $\mathfrak{C}(E \times E)$ cannot

- If E is a CM elliptic curve such that the order of $[E] \in \mathrm{Cl}(\mathrm{End}(E))$ is greater than 2, then

$$[\mathfrak{C}(E \times E)]^2 = [E]^4 \neq [E]^2$$

- This means that $j(\mathfrak{C}(E \times E)) \notin \mathbb{Q}(j(E)) = L$ by our field of definition condition

- It is still true that both $j(E)$ and $j(\mathfrak{C}(E \times E)) \in L_{\mathrm{End}(E)}$, the ring class field

Field of definition

Isomorphism classes of abelian surfaces

- To determine the "field of definition" of an abelian surface of maximal Picard rank $\mathscr{A} = E_1 \times E_2$ we must determine when $\mathscr{A} \cong \mathscr{A}'$ for another abelian surface of maximal Picard rank

- **Shioda and Mitani** proved that for $E_1 \times E_2$ of maximal Picard rank, we have $E_1 \times E_2 \cong E_3 \times E_4$ if and only if
  (i) $\mathrm{End}(E_1) \cap \mathrm{End}(E_2) = \mathrm{End}(E_3) \cap \mathrm{End}(E_4)$
  (ii) $\mathrm{End}(E_1) + \mathrm{End}(E_2) = \mathrm{End}(E_3) + \mathrm{End}(E_4)$
  (iii) $\mathfrak{C}(E_1 \times E_2) \cong \mathfrak{C}(E_3 \times E_4)$

- So in particular, if $\mathrm{End}(E_1) = \mathrm{End}(E_2) = R$ then $E_1 \times E_2 \cong E_3 \times E_4$ if and only if both
  (1) $R = \mathrm{End}(E_3) = \mathrm{End}(E_4)$
  (2) $\mathfrak{C}(E_1 \times E_2) \cong \mathfrak{C}(E_3 \times E_4)$, or $[E_1][E_2] = [E_3][E_4] \in \mathrm{Cl}(R)$

Field of definition

Interesting examples, part 2

- If E is a CM elliptic curve such that the order of $[E] \in \mathrm{Cl}(\mathrm{End}(E))$ is greater than 2, then $j(\mathfrak{C}(E \times E)) \notin \mathbb{Q}(j(E))$
- However, by the **Shioda-Mitani condition**, $E \times E$ is isomorphic to various other products $E_1 \times E_2$ with

$$\mathrm{End}(E_1) = \mathrm{End}(E_2) = \mathrm{End}(E) \text{ and } [E]^2 = [E_1][E_2]$$

- If $[E] \in \mathrm{Cl}(\mathrm{End}(E))$ has order 4, then

$$E \times E \cong \mathbb{C}/\mathrm{End}(E) \times \mathfrak{C}(E \times E)$$

- By our field of definition condition,

$$\mathbb{Q}(j(\mathfrak{C}(E \times E))) = \mathbb{Q}(j(\mathbb{C}/\mathrm{End}(E)))$$

- This means $E \times E$ and $\mathfrak{C}(E \times E)$ can both be defined over a field smaller than the ring class field

Conclusion

In summary

- For $X \cong E_1 \times E_2 \times \ldots \times E_n$ pairwise isogenous CM elliptic curves with $\mathrm{End}(E_i) = R_i$, then $\mathfrak{C}_n(X)$ is an elliptic curve with CM by $R_1 + \ldots + R_n \subseteq K = \mathrm{End}(E_i) \otimes_{\mathbb{Z}} \mathbb{Q}$

- For $m < n$ we have

$$\mathfrak{C}_m(X) \cong \prod_{i_1 < i_2 < \ldots < i_m} \mathfrak{C}_m\left(E_{i_1} \times \ldots \times E_{i_m}\right)$$

  which is also a complex abelian variety of maximal Picard number

- We gave examples of some abelian surfaces $\mathscr{A}$ over a number field L so that $j(\mathfrak{C}(\mathscr{A}_{\mathbb{C}})) \notin L$

- This means though $\mathfrak{C}(\mathscr{A}_{\mathbb{C}})$ is algebraic (it is an abelian variety), the construction of $\mathfrak{C}(\mathscr{A}_{\mathbb{C}})$ is **not quite algebraic**

Conclusion

Future work

- For X a complex abelian variety of maximal Picard number and dimension n, $\mathfrak{C}_{n-1}(X)$ is also a complex abelian variety of maximal Picard number and dimension n
  - Does the $\mathfrak{C}_{n-1}$ action on the set of such X have finite orbits?
  - Does the action have fixed points?
- Beauville computed $\mathfrak{C}(S)$ up to isogeny for some surfaces of maximal Picard number that are not abelian surfaces
  - Can we compute these $\mathfrak{C}(S)$ up to isomorphism?
  - Are there complex surfaces such that $\mathfrak{C}(S)$ is a complex torus but not an abelian variety?

Conclusion

Thanks and acknowledgments

- This talk is based on joint work with my NSF postdoc mentor Max Lieblich

- I would also like to thank Lalita Devadas for useful conversations about lattices

- This talk is based on work supported by the National Science Foundation under Award No. 2102960

Conclusion

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