Ceresa cycles, covers of curves, and unlikely intersections

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Outline

Algebraic cycles on abelian varieties

X/k: a smooth projective geometrically integral curve over k

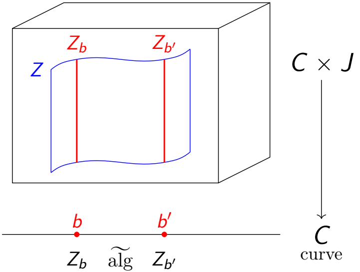
J: Jacobian of X

$\mathrm{Z}_1(J)$: Free abelian group on dimension 1 subvarieties of J

$\mathrm{Z}_1(J)$ is a very large group!

Notions of equivalence: rational or algebraic or homological.

Filtration (*) by cycles trivial under each:

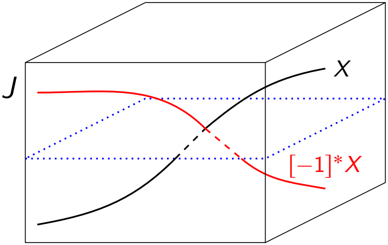


$$\underset{C=\mathbb{P}^1}{\mathrm{Z}_{1,\mathrm{rat}}(J)} \subset \underset{\langle Z_b - Z_{b'} \rangle}{\mathrm{Z}_{1,\mathrm{alg}}(J)} \subset \mathrm{Z}_{1,\mathrm{hom}}(J) \subset \mathrm{Z}_1(J)$$

The image of the Abel-Jacobi map

$$i_b : X \to J$$
$$x \mapsto [x - b]$$

is a cycle $[X] \in Z_1(J)$.



Definition

The Ceresa cycle $\mathrm{Cer}(X, b)$ *in* $\mathrm{Z}_1(J)$ *is the canonical cycle*

$$\mathrm{Cer}(X, b) := [X] - [-1]^*[X].$$

Since $[-1]^*$ is trivial on homology, $\mathrm{Cer}(X, b) \in \mathrm{Z}_{1,\mathrm{hom}}(J)$.

Question: How deep in the filtration $(*)$ does $\mathrm{Cer}(X, b)$ lie?

Where is the Ceresa cycle rationally trivial?

Hyperelliptic curves have trivial Ceresa cycle.

Theorem (Ceresa '83)

Let $g \geqslant 3$. Then a very general $X/\mathbb{C}$ of genus g has infinite order Ceresa cycle.

Strengthened by Hain, Zhang & Gao, Kerr & Tayou in 2024.

Upshot: For $g \geqslant 3$, the Ceresa–torsion locus in $\mathcal{M}_g$ is a (non-explicit!) countable union of closed subvarieties.

Question: Where does Y_t intersect the Ceresa–torsion locus?

Easier Q: Where does a section P_t of an elliptic fibration E_t become torsion?

Outline

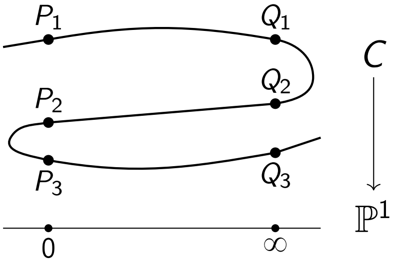
For C a genus g curve over a field k, the Jacobian of C is a g-dimensional abelian variety also defined over k.

Elements of the abstract group $\mathrm{Jac}(C)$ are degree 0 divisors

$$\sum_{P \in C} n_P P \text{ with } \sum_{P \in C} n_P = 0$$

modulo rational equivalence:

$$P_1 + P_2 + P_3 = Q_1 + Q_2 + Q_3.$$



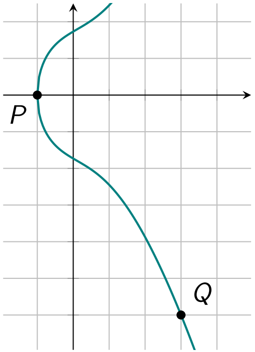
When $g = 1$, the Jacobian of C is an elliptic curve.

Canonical height: A tool for testing if a point is torsion

Elements of the Jacobian are points of a variety:
they have coordinates!

For $E/\mathbb{Q}$, we get a canonical height $\hat{h} : E(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geqslant 0}$ such that

$$\hat{h}(P) = 0 \iff P \text{ torsion.}$$



$E : y^2 = x^3 + 2x + 3$

$P = (-1, 0)$ has height 0 and order 2.

$Q = (3, -6)$ has height ≈ 1.45 and infinite order.

Testing independence of points on E using heights

For $E/\mathbb{Q}$, and points $P_1, \ldots, P_n \in E(\overline{\mathbb{Q}})$ we can test their independence with the height pairing matrix

$$\begin{pmatrix} \langle P_1, P_1\rangle & \ldots & \langle P_n, P_1\rangle \\ & \ldots & \\ \langle P_1, P_n\rangle & \ldots & \langle P_n, P_n\rangle \end{pmatrix}$$

for the bilinear height pairing

$$\langle P, Q\rangle = \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q).$$

Key: The determinant of the matrix is nonzero if and only if the points are linearly independent.

Consider the elliptic curve

$$E : x^3 - 3xy + y^2 + 9y + 8 = 0.$$

and the points

$$P_1 = (0, -1) - (\omega - \omega^2, -1) + (-2, 0) - (-2\omega, 0)$$
$$P_2 = (0, -1) - (\omega^2 - \omega, -1) + (-2, 0) - (-2\omega^2, 0).$$

MAGMA: det of the height pairing matrix for P_1, P_2 is $\sim$47.72
$\implies P_1, P_2$ are linearly independent!

Theorem (Laga–Shnidman '23)

The Ceresa cycle of the curve $X_t : y^3 = x^4 + 2tx^2 + 1$ *is torsion if and only if the point* $(\sqrt[3]{t^2 - 1}, t)$ *on the elliptic curve* $y^2 = x^3 + 1$ *is torsion.*

Proof technique: Chow motives

This means $\mathrm{Cer}(X_t)$ is torsion for ∞ values of $t \in \mathbb{C}$
(but a Northcott property holds).

Question: Can we find a family of curves Y_t such that the Ceresa cycle is torsion for finitely many $t \in \mathbb{C}$, using classical techniques?

Let $Y_t \to C_t$ a double cover branched above $(1, -1), (-t, 0)$ for

$$C_t : y^3 + z^3 + y^2 z^2 + (t^3 + 1)yz + t^3 = 0.$$

Theorem 1 (Bhatnagar–Devadas–D'Nelly-Warady–Srinivasan)

The Ceresa cycle $\mathrm{Cer}(Y_t)$ *is torsion for finitely many* $t \in \mathbb{C}$.

Key: The family C_t has automorphism group S_3 with involutions

$$\sigma_i : (y, z) \mapsto (\omega^i z, \omega^{-i} y) \text{ for } i = 1, 2, 3.$$

The quotients C_t/σ_i are isomorphic to the elliptic curve

$$E_t : x^3 - 3xy + y^2 + (t^3 + 1)y + t^3 = 0.$$

Theorem 2 (Bhatnagar–Devadas–D'Nelly-Warady–Srinivasan)

Let G be a finite group. Let $\mathcal{M}_G$ denote the space of genus g curves Y admitting automorphisms by G.

If $g(Y/G) \geqslant 2$ and $Y \to Y/G$ is ramified at at least one point, then the very general curve in $\mathcal{M}_G$ has infinite order Ceresa cycle.

Our dimension reduction technique for both theorems:
Intersection theory of algebraic cycles using covers $Y \to C$.

Dimension reduction: Shadows of the Ceresa cycle

Definition (Ellenberg–Logan–Srinivasan '24)

Let $\phi\colon C \to C'$ be a separable degree d cover of curves with ramification divisor R_ϕ. The relative canonical shadow in $\mathrm{Pic}^0(C)$ is

$$D_\phi := d(2g_{C'} - 2)R_\phi - \deg(R_\phi)\phi^*(K_{C'}) + 2(dR_\phi - \phi^*\phi_* R_\phi)).$$

When ϕ is Galois, $2(dR_\phi - \phi^*\phi_* R_\phi)) = 0$ (since $\phi^*\phi_* = [d]$).

When C' is an elliptic curve, $d(2g_{C'} - 2)R_\phi - \deg(R_\phi)\phi^*(K_{C'}) = 0$ so $D_\phi = 2(dR_\phi - \phi^*\phi_* R_\phi)$.

Theorem (E–L–S '24)

If D_ϕ is infinite order, then the Ceresa cycle of C has infinite order.

Proof of Theorem 1: $\mathrm{Cer}(Y_t)$ is torsion for finitely many t

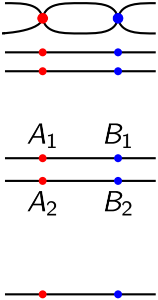
$g = 4$ $\quad Y_t$

ϕ $\quad \downarrow \pi$

$g = 3$ $\quad C_t$

$\downarrow \psi$

$g = 1$ $\quad E_t$



π is branched above A_1, B_1.

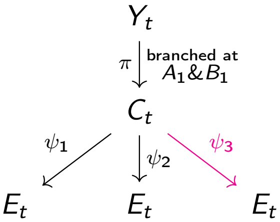
$\phi = \psi \circ \pi \colon Y_t \to E_t$ is <u>not</u> Galois.

$\pi_*(D_\phi) = 4(A_1 - A_2 + B_1 - B_2)$

Goal: Show $\pi_*(D_\phi)$ (hence D_ϕ) has infinite order.

$\psi_* \pi_*(D_\phi) = 0$, but C_t admits other maps to E_t!

Thm 1 Proof cont'd: $\mathrm{Cer}(Y_t)$ is torsion for finitely many t



Recall C_t has 3 involutions

$$\sigma_i : (y, z) \mapsto (\omega^i z, \omega^{-i} y)$$

with quotients $C_t/\sigma_i \cong E_t$ for

$$E_t : x^3 - 3xy + y^2 + (t^3+1)y + t^3 = 0.$$

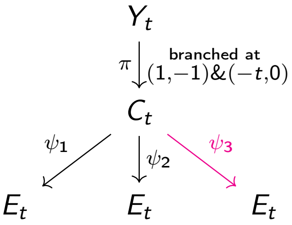
We get three shadows! One for each of the maps $\psi_i \circ \pi \colon Y_t \to E_t$.

Push–forward along $\psi_3 \circ \pi$ gives sections P_i of E_t:

$\psi_{3,*}\pi_*(D_{\psi_i \circ \pi}) = 4P_i$ for

$$P_i = \psi_3(A_1) - \psi_3(\sigma_i(A_1)) + \psi_3(B_1) - \psi_3(\sigma_i(B_1)).$$

Thm 1 Proof: Reduction to an unlikely intersections result



We have two sections
$P_1(t), P_2(t) \in E_t$ such that

$\mathrm{Cer}(Y_t)$ torsion $\implies$
$D_{\psi_i \circ \pi}$ torsion $\implies$
$P_i(t)$ torsion for both $i = 1, 2$.

By our earlier height pairing matrix computation,
$P_1(2), P_2(2)$ are linearly independent points of $E_2 \implies$
$P_1(t), P_2(t)$ are linearly independent sections of E_t!

Masser–Zannier unlikely intersections theorem $\implies$
$P_1(t), P_2(t)$ are simultaneously torsion for finitely many $t \in \mathbb{C}$. $\square$

Theorem 2

Let G be a finite group. Let $\mathcal{M}_G$ denote the space of genus g curves Y admitting automorphisms by G.

If $g(Y/G) \geqslant 2$ and $Y \to Y/G$ is ramified at at least one point, then the very general curve in $\mathcal{M}_G$ has infinite order Ceresa cycle.

Proof: $\pi : Y \to Y/G$ is Galois, so we may compute the shadow:

$$\pi_*(D_\pi) = (\#G)\left((2g_{Y/G} - 2)\pi_*(R_\pi) - (\deg R_\pi)K_{Y/G}\right).$$

Manin–Mumford theorem $\implies$ For a very general choice of branch points, $\pi_*(D_\pi)$ (hence $\mathrm{Cer}(Y)$) has infinite order. $\square$

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