Higher-weight Jacobians

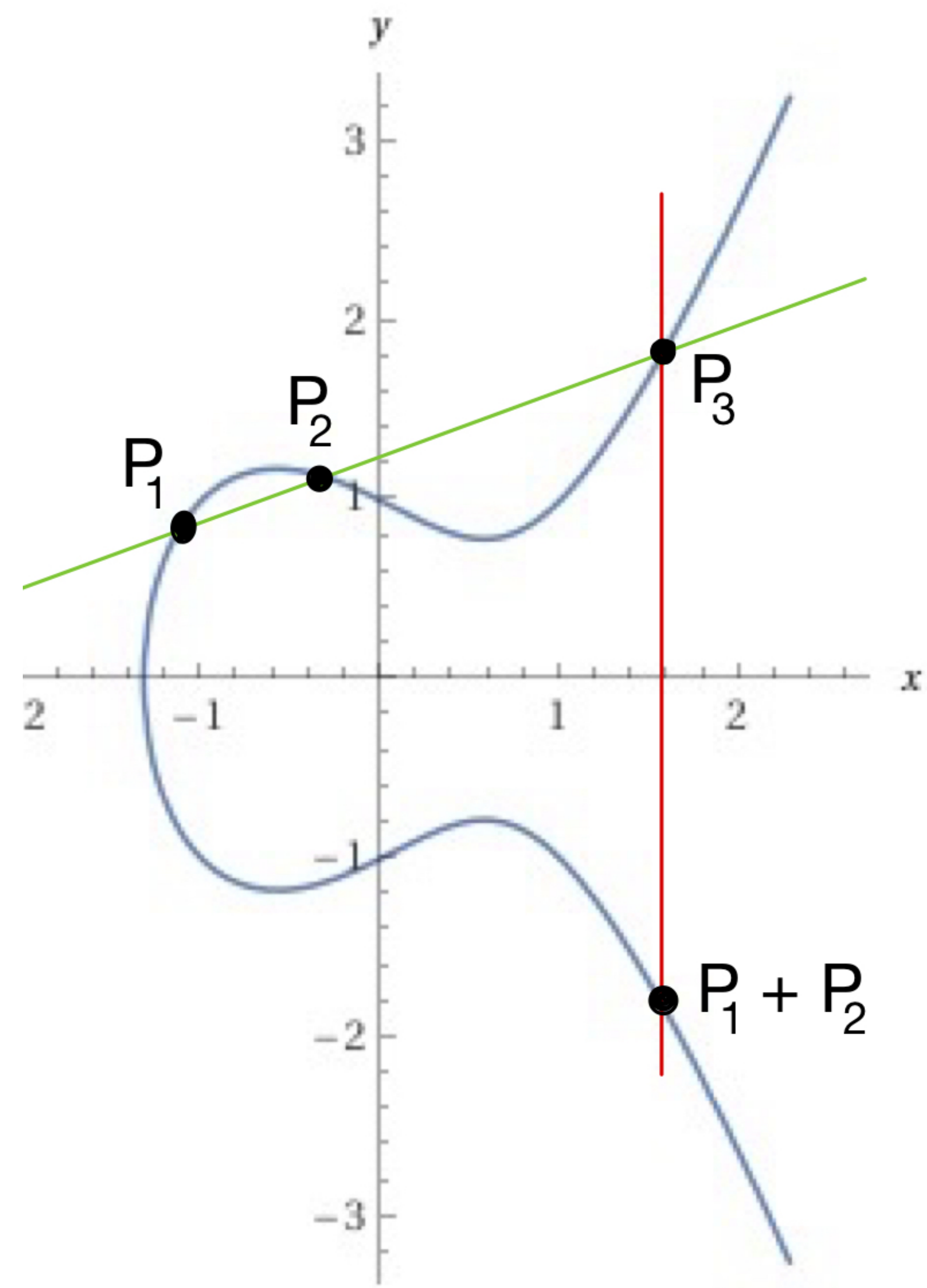
Generalizing the notion of adding points on a curve

Work by Sheela Devadas and Max Lieblich

Motivation

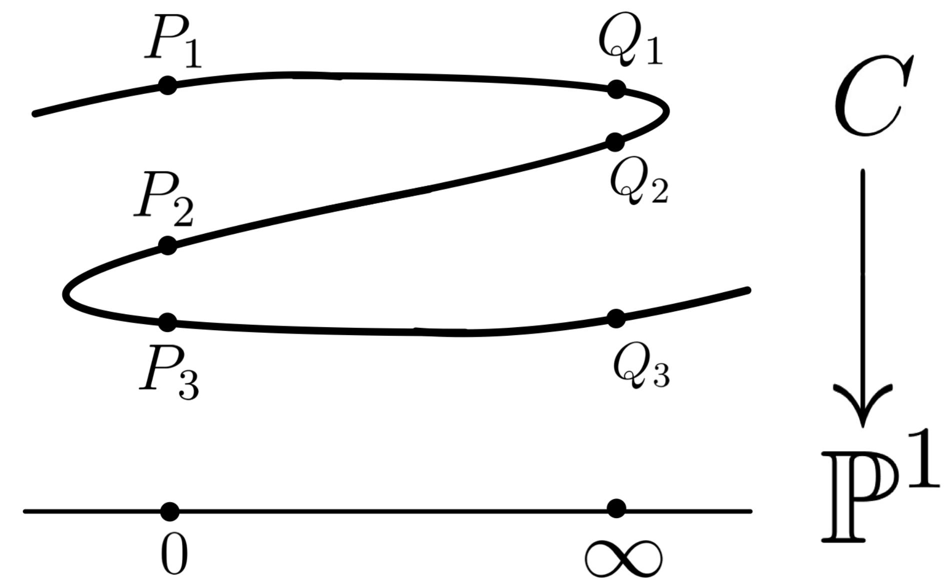
Elliptic curves in algebraic geometry

- In algebraic geometry we study **algebraic varieties**, which are the set of solutions of a system of polynomial equations
- An **elliptic curve** is the solution set of an equation $y^2 = x^3 + ax + b$
- In fact elliptic curves are **abelian varieties**, since we can “add” points, giving it the structure of an abelian group
- These important objects of number theory have applications in cryptography, integer factorization, and primality proving



Motivation

Jacobian of a complex curve



- For more general curves we cannot add points as with an elliptic curve
- Instead, we can embed the curve C in a higher-dimensional abelian variety known as the **Jacobian variety** $\mathrm{Jac}(C)$
- Elements of $\mathrm{Jac}(C)$ are **degree 0 divisors**—abstract "combinations of points"

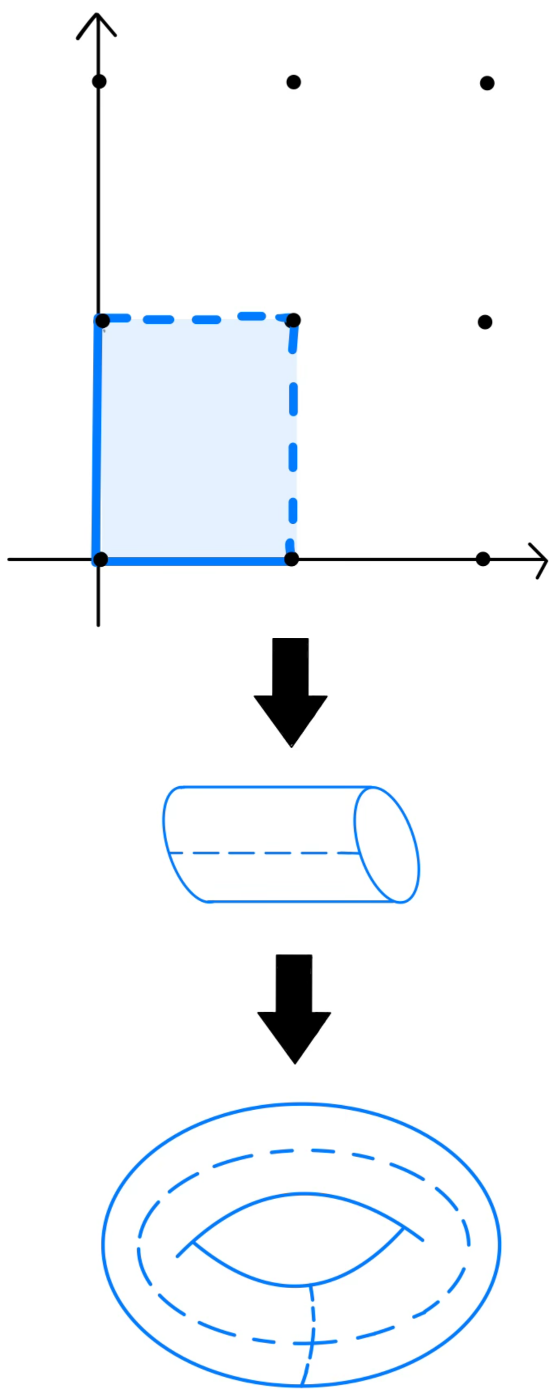
$$\sum_{P \in C} n_P P \text{ such that } \sum_{P \in C} n_P = 0$$

  modulo the relation of rational equivalence ($P_1 + P_2 + P_3 = Q_1 + Q_2 + Q_3$)

- Geometrically, we can give this abelian group the structure of a variety
- In addition to this algebraic definition, we can give an analytic construction of the Jacobian variety as a complex torus

Motivation

Complex abelian varieties and tori



- A **complex torus** $\mathbb{C}^d/\Gamma$ is the quotient of a complex vector space $\mathbb{C}^d$ by a discrete subgroup or lattice Γ

- Any complex abelian variety can be realized as a complex torus which admits an algebraic structure

- For $d = 1$ every torus is algebraic, so $\mathbb{C}/\Lambda$ is always isomorphic to an elliptic curve over $\mathbb{C}$

- This is not true in higher dimensions!

Motivation

Analytic construction of the Jacobian

- A variety V is equipped with a **topology**—a collection of subsets termed as open subsets—and a **structure sheaf** $\mathcal{O}_V$ of rings, associating a ring to each open subset compatibly with restriction of subsets

- Given a complex curve C we have a short exact sequence of sheaves (called the **exponential sequence**)

$$\mathbb{Z} \hookrightarrow \mathcal{O}_C \overset{\exp}{\twoheadrightarrow} \mathcal{O}_C^\times$$

- This gives rise to a long exact sequence of sheaf cohomology groups

$$0 \to \mathrm{H}^0(C, \mathbb{Z}) \to \mathrm{H}^0(C, \mathcal{O}_C) \to \mathrm{H}^0(C, \mathcal{O}_C^\times) \to \mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathcal{O}_C) \to \dots$$

- The cokernel of $\mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathcal{O}_C)$ is a **complex torus** $\mathbb{C}^g/\Gamma$

- This torus gives an **analytic construction** of the Jacobian variety

- This torus is algebraic, which makes it an **abelian variety**

Motivation

Higher dimensional complex varieties

- The Jacobian of a complex curve can be generalized to give an algebraic complex torus structure to the cokernel of $\mathrm{H}^1(X, \mathbb{Z}) \to \mathrm{H}^1(X, \mathcal{O}_X)$ for higher-dimensional X

- If X is smooth and projective, this construction is a complex torus because the image of $\mathrm{H}^1(X, \mathbb{Z})$ in $\mathrm{H}^1(X, \mathcal{O}_X)$ is a **discrete subgroup** of the complex vector space $\mathrm{H}^1(X, \mathcal{O}_X)$

- Is the cokernel of $\mathrm{H}^m(X, \mathbb{Z}) \to \mathrm{H}^m(X, \mathcal{O}_X)$ also a complex torus for $m > 1$?

- If $m > 1$,
  1. the image of $\mathrm{H}^m(X, \mathbb{Z})$ in $\mathrm{H}^m(X, \mathcal{O}_X)$ is **not necessarily discrete**, so the cokernel of $\mathrm{H}^m(X, \mathbb{Z}) \to \mathrm{H}^m(X, \mathcal{O}_X)$ is **not** necessarily a complex torus;
  2. even if it is a complex torus, it is **not** necessarily algebraic

Motivation

Degree 2: the Brauer-Jacobian

- The cokernel $\mathfrak{C}(X)$ of $\mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{O}_X)$ is a complex torus exactly when the kernel has maximal rank—meaning X is "very algebraic" (has many line bundles)

- We say that X which satisfies the above has **maximal Picard number**

- The torsion subgroup of $\mathfrak{C}(X)$ is isomorphic to the Brauer group $\mathrm{Br}(X)$, so we call $\mathfrak{C}(X)$ the **Brauer-Jacobian**

- $\mathfrak{C}(X)$ has been studied by Beauville '14, Shioda & Mitani '74, among others

- When X is an abelian variety, $\mathfrak{C}(X)$ has been computed up to **isogeny** (a surjective morphism with finite kernel)

- Furthermore, in this case $\mathfrak{C}(X)$ is algebraic (an abelian variety)

Motivation

Products of elliptic curves

- Recall: an **elliptic curve** is a 1-dimensional abelian variety, which is often described as the solution set of an equation $y^2 = x^3 + ax + b$

- We say the elliptic curve has **complex multiplication (CM)** if it has more endomorphisms than just multiplication by $\mathbb{Z}$

Theorem (Schoen). A complex abelian variety X of maximal Picard rank is (not uniquely) isomorphic to a **product of pairwise isogenous CM elliptic curves**

$$X \cong E_1 \times E_2 \times \ldots \times E_n$$

- In the $n = 2$ case, Shioda and Mitani show that $\mathfrak{C}(E_1 \times E_2)$ is an elliptic curve that is isogenous to both E_1 and E_2

Our results

- Theorem (Schoen). A complex abelian variety X of maximal Picard rank is (not uniquely) isomorphic to a **product of pairwise isogenous CM elliptic curves**

$$X \cong E_1 \times E_2 \times \ldots \times E_n$$

- In the $n = 2$ case, Shioda and Mitani show that $\mathfrak{C}(E_1 \times E_2)$ is an elliptic curve that is isogenous to both E_1 and E_2

- My work with M. Lieblich uses this concrete description of X to compute $\mathfrak{C}(X)$ up to **isomorphism**, via number theory

- We proved that for such X, the cokernel of $\mathrm{H}^m(X, \mathbb{Z}) \to \mathrm{H}^m(X, \mathcal{O}_X)$ for **all** $m \leq n$ is also a complex abelian variety of maximal Picard number, which we call the **weight** m **Jacobian** $\mathfrak{C}_m(X)$

- We also proved conditions on the **field of definition** for $\mathfrak{C}_2(E_1 \times E_2)$ in certain cases, and we plan to prove similar conditions in the general case

Roadmap

- **Number theory**
  - Elliptic curves over $\mathbb{C}$ and lattices
  - Complex multiplication and the ideal class group
- Computing higher-weight Jacobians
- Fields of definition
  - Issues with extending to fields other than $\mathbb{C}$
- Conclusions and future work

Number theory

Lattices in the complex numbers

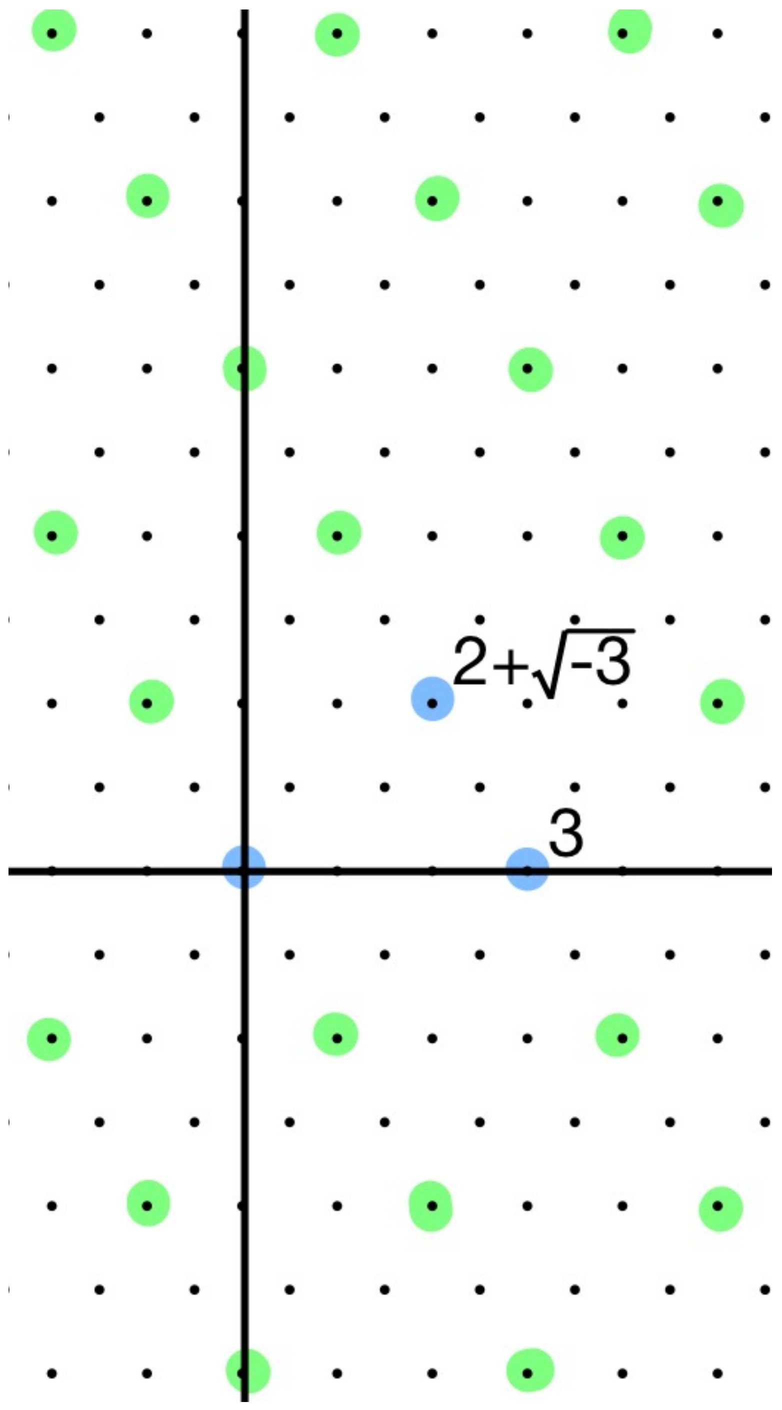
- Any complex elliptic curve E is isomorphic as a Lie group to a torus $\mathbb{C}/\Gamma$ for a discrete subgroup or **lattice** $\Gamma \subseteq \mathbb{C}$

- Any lattice in $\mathbb{C}$ can be written as the set of $\mathbb{Z}$-linear combinations of some $v, w \in \mathbb{C}$ with $v/w \notin \mathbb{R}$:
$$\Gamma = \mathbb{Z}v + \mathbb{Z}w = \langle v, w \rangle$$

- An **isomorphism** of elliptic curves $E \cong \mathbb{C}/\Gamma, E' \cong \mathbb{C}/\Gamma'$ corresponds to a **homothety** (rotation/scaling) of lattices, which is some $\alpha \in \mathbb{C}$ with $\alpha\Gamma = \Gamma'$

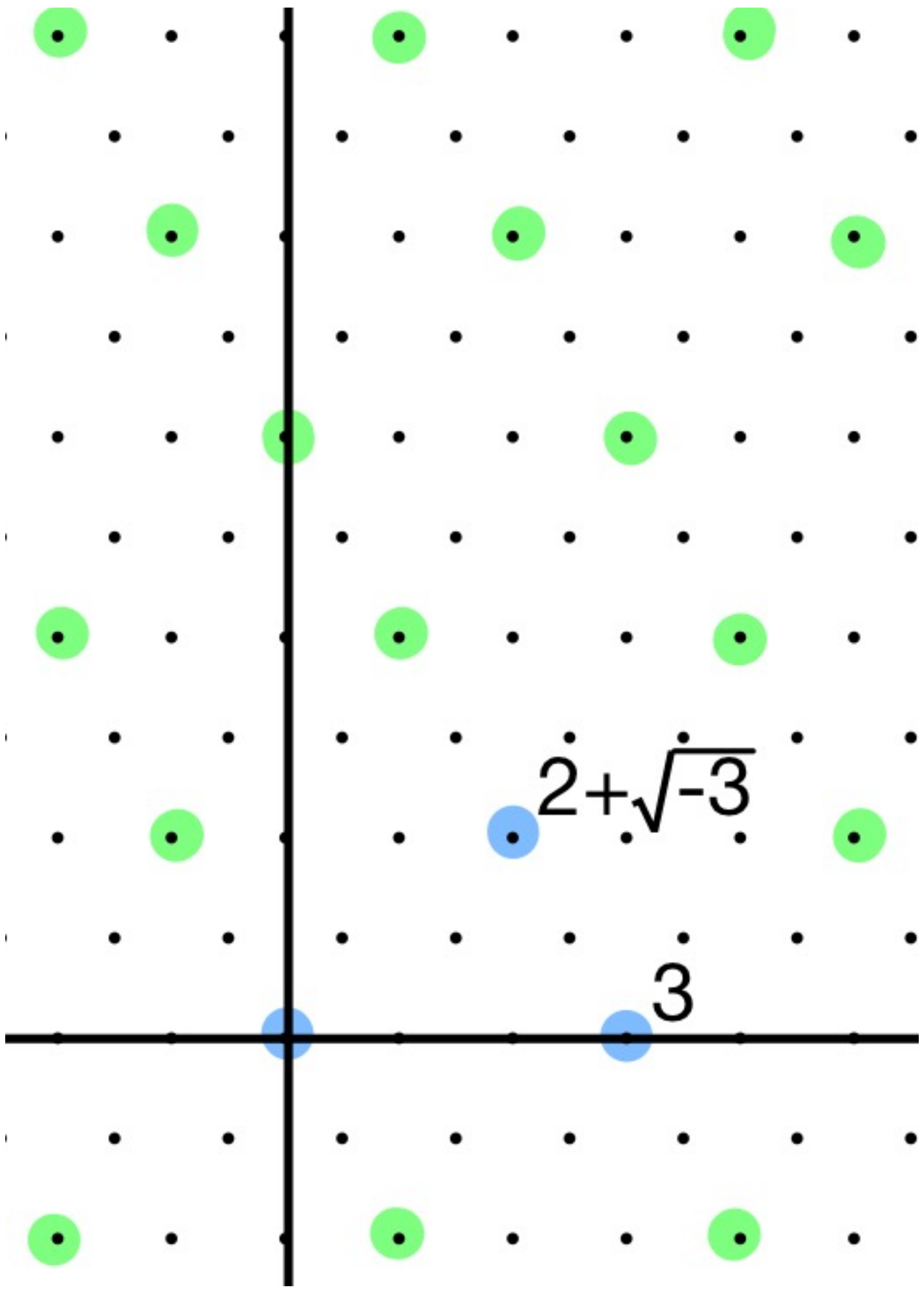
- An isogeny of elliptic curves corresponds to $\alpha \in \mathbb{C}$ with $\alpha\Gamma \subseteq \Gamma'$

- Two lattices are **isogenous** if one is **homothetic to a sublattice** of the other



Number theory

Complex multiplication



- The **endomorphism ring** of a lattice is the ring of isogenies to itself

$$\mathrm{End}(\Gamma) = \{\alpha \in \mathbb{C} : \alpha\Gamma \subseteq \Gamma\}$$

- If $\mathrm{End}(\Gamma)$ is larger than $\mathbb{Z}$ we say that the lattice/elliptic curve has **complex multiplication (CM)** by $\mathrm{End}(\Gamma)$

- For $\Lambda = \langle 3, 2+\sqrt{-3}\rangle$ we have

$$\mathrm{End}(\Lambda) = \mathbb{Z}[3\sqrt{-3}]$$

- Λ is homothetic to $3\Lambda = \langle 9, 6+3\sqrt{-3}\rangle \subseteq \mathbb{Z}[3\sqrt{-3}]$, which is an ideal of $\mathbb{Z}[3\sqrt{-3}]$

Number theory

CM elliptic curves and the class group

- If $E \cong \mathbb{C}/\Gamma$ has CM, then Γ is homothetic to an ideal of $\mathrm{End}(\Gamma) = \mathrm{End}(E)$
- Think of Γ as a (fractional) ideal of $R = \mathrm{End}(\Gamma)$, and homothety as multiplication by an element of R—or by a principal ideal
- The **ideal class group** $\mathrm{Cl}(R)$ is the group of (invertible) fractional ideals of R modulo principal ideals (R is the identity element)
- Homothety classes of lattices with CM by R correspond to elements of $\mathrm{Cl}(R)$
- We use $[E] = [\Gamma] \in \mathrm{Cl}(R)$ for the group element corresponding to the homothety/isomorphism class of $E \cong \mathbb{C}/\Gamma$

Roadmap

- Number theory
- **Computing higher-weight Jacobians**
  - Lattices and the Brauer-Jacobian (2-Jacobian)
  - Self-product example $E \times E$
  - 2-Jacobian of an abelian surface
  - The n-Jacobian of an abelian n-fold
  - Computing the m-Jacobian
- Fields of definition
  - Issues with extending to fields other than $\mathbb{C}$
- Conclusions and future work

Computing higher-weight Jacobians

Isogeny of CM lattices

- Recall: a complex abelian variety X of maximal Picard number is isomorphic to a **product of pairwise isogenous CM elliptic curves** (Schoen)

$$X \cong E_1 \times E_2 \times \ldots \times E_n$$

- What does this mean in terms of lattices?

- If Γ is a CM lattice with $R = \mathrm{End}(\Gamma)$, then the **endomorphism algebra** $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$

- In fact, R is a subring of the endomorphism algebra $K = R \otimes_{\mathbb{Z}} \mathbb{Q}$

- Two CM elliptic curves E_1, E_2 are isogenous **if and only if** the endomorphism algebras are the same

Computing higher-weight Jacobians

Lattices and the 2-Jacobian

- Shioda and Mitani found that $\mathfrak{C}(E_1 \times E_2)$ is a CM elliptic curve **isogenous** to both E_1, E_2

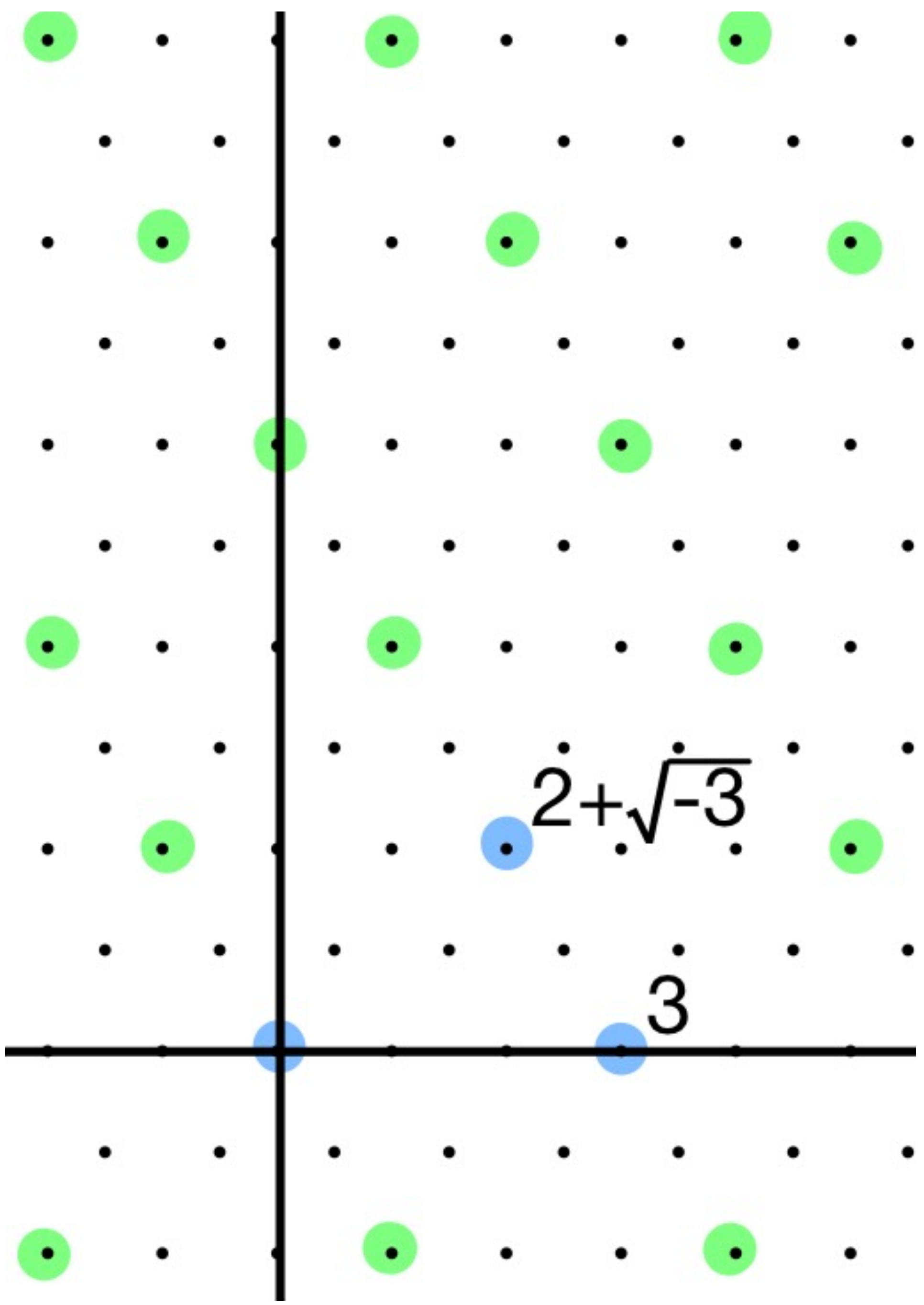
- To compute $\mathfrak{C}(E_1 \times E_2)$ up to **isomorphism**, we'll find $R = \mathrm{End}(\mathfrak{C}(E_1 \times E_2))$ and an element in $\mathrm{Cl}(R)$ for the isomorphism class $[\mathfrak{C}(E_1 \times E_2)]$

- It's known that if $E_i \cong \mathbb{C}/\langle v_i, w_i \rangle$ and $v_1v_2, v_1w_2, w_1v_2, w_1w_2 \in \mathbb{C}$ span a lattice Γ then $\mathfrak{C}(E_1 \times E_2) \cong \mathbb{C}/\Gamma$

Computing higher-weight Jacobians

Self-product example

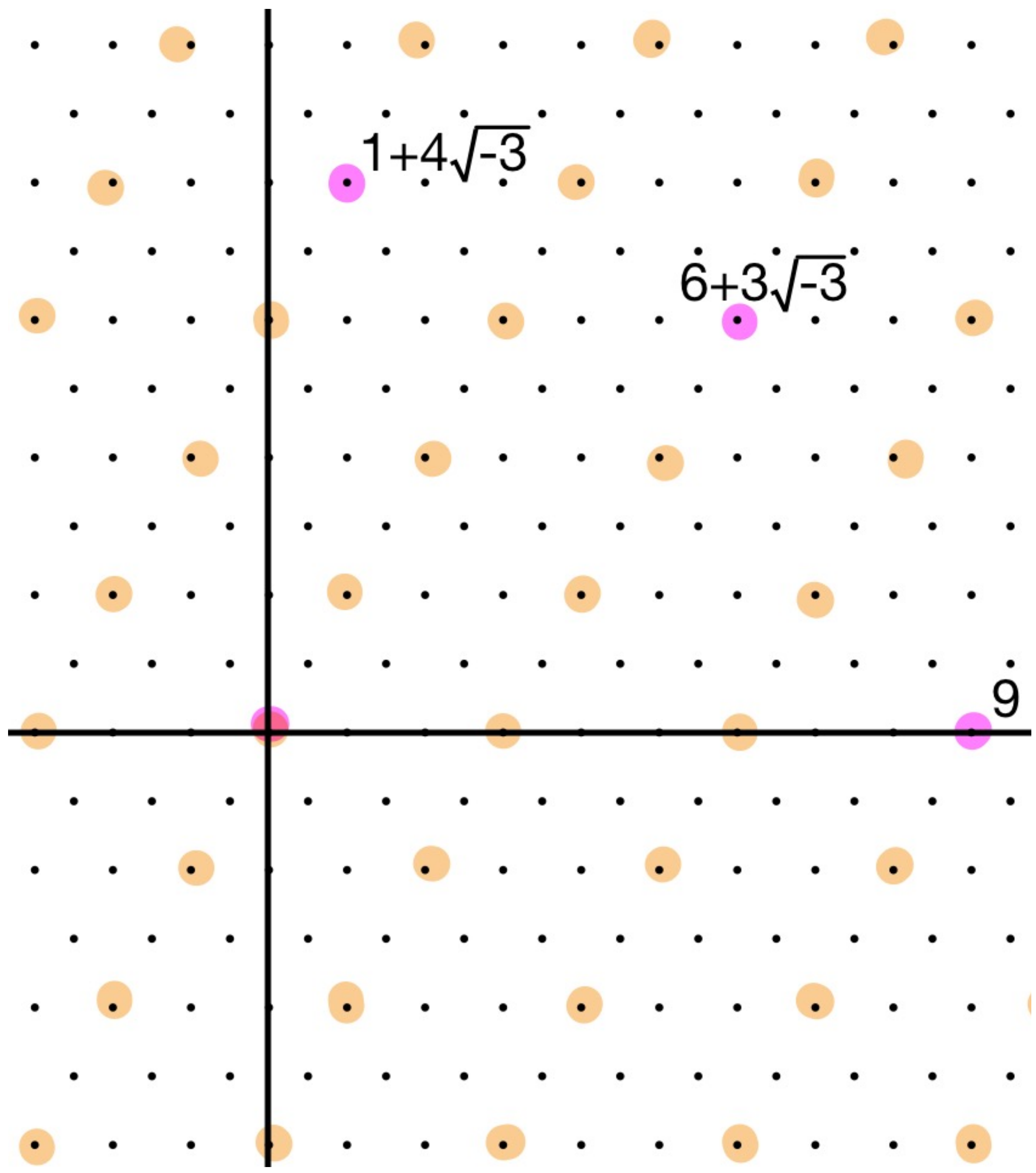
- Consider the elliptic curve $E = \mathbb{C}/\Lambda$ corresponding to the lattice $\Lambda = \langle 3, 2 + \sqrt{-3} \rangle$
- $\mathfrak{C}(E \times E)$ is isomorphic to an elliptic curve $E' \cong \mathbb{C}/\Lambda'$
- Λ' is spanned by
  - $3^2 = 9$
  - $3 \cdot (2 + \sqrt{-3}) = 6 + 3\sqrt{-3}$
  - $(2 + \sqrt{-3})^2 = 1 + 4\sqrt{-3}$



Computing higher-weight Jacobians

Self-product example

- $\{9, 6+3\sqrt{-3}, 1+4\sqrt{-3}\}$ span the lattice $\Lambda' = \langle 3, 1+\sqrt{-3}\rangle$
- Hence $\mathfrak{C}(E \times E)$ is isomorphic to $E' = \mathbb{C}/\Lambda'$
- Λ, Λ' are **not homothetic**, so $E \not\cong E' \cong \mathfrak{C}(E \times E)$
- However, they are isogenous; both have endomorphism ring $\mathbb{Z}[3\sqrt{-3}]$



Computing higher-weight Jacobians

2-Jacobian of an abelian surface

- The easiest case to compute the isomorphism class $[\mathfrak{C}(E_1 \times E_2)]$ is when $\mathrm{End}(E_1) = \mathrm{End}(E_2) = R$

- In that case, $\mathfrak{C}(E_1 \times E_2)$ **also has CM by** R and
$$[\mathfrak{C}(E_1 \times E_2)] = [E_1][E_2] \in \mathrm{Cl}(R)$$

- More generally, if $\mathrm{End}(E_i) = R_i$, then we can show $\mathfrak{C}(E_1 \times E_2)$ **has CM by** $R_0 := R_1 + R_2 \subseteq K = R_i \otimes_{\mathbb{Z}} \mathbb{Q}$

- Letting $E_i \cong \mathbb{C}/\Gamma_i$ we find that $\Gamma_i \otimes_{R_i} R_0$ is a lattice with CM by R_0 and
$$[\mathfrak{C}(E_1 \times E_2)] = [\Gamma_1 \otimes_{R_1} R_0][\Gamma_2 \otimes_{R_2} R_0] \in \mathrm{Cl}(R_0)$$

Computing higher-weight Jacobians

The *n*-Jacobian of an abelian *n*-fold

- Recall: a complex abelian variety X of maximal Picard number is isomorphic to a **product of pairwise isogenous CM elliptic curves** (Schoen)

$$X \cong E_1 \times E_2 \times \ldots \times E_n$$

- Since X has maximal Picard rank, $\mathfrak{C}(X)$ is a complex torus

- For $m \geq 2$, the cokernel of $\mathrm{H}^m(X, \mathbb{Z}) \to \mathrm{H}^m(X, \mathcal{O}_X)$ is **also a complex torus,** the m-Jacobian $\mathfrak{C}_m(X)$

- If $n = m$, then we find similarly that $\mathfrak{C}_n(E_1 \times \ldots \times E_n)$ is an elliptic curve with CM by R, where R is

$$R_1 + \ldots + R_n \subseteq K = R_i \otimes_{\mathbb{Z}} \mathbb{Q}, \text{ for } R_i = \mathrm{End}(E_i)$$

- We can compute $\mathfrak{C}_n(X)$ up to isomorphism as

$$[\mathfrak{C}_n(X)] = \prod_{i=1}^{n} [\Gamma_i \otimes_{R_i} R] \in \mathrm{Cl}(R), \text{ for } \Gamma_i \text{ such that } E_i \cong \mathbb{C}/\Gamma_i$$

Computing higher-weight Jacobians

Computing the *m*-Jacobian

- For $m < n$, we find that the ***m*-Jacobian** of X is

$$\mathfrak{C}_m(X) \cong \prod_{i_1 < i_2 < \ldots < i_m} \mathfrak{C}_m\left(E_{i_1} \times \ldots \times E_{i_m}\right)$$

- Each $\mathfrak{C}_m(E_{i_1} \times \ldots \times E_{i_m})$ is a **CM elliptic curve isogenous to the** E_i

- $\mathfrak{C}_m(X)$ is a product of pairwise isogenous CM elliptic curves

Theorem (Devadas-Lieblich). If X is a complex abelian variety of maximal Picard rank, the m-Jacobian $\mathfrak{C}_m(X)$ for any $2 \leq m \leq \dim(X)$ is also a complex abelian variety of maximal Picard rank.

- In particular, $\mathfrak{C}_{n-1}(X)$ is a complex abelian variety of the same dimension n

Roadmap

- Number theory
- Computing higher-weight Jacobians
- **Fields of definition**
  - Issues with extending to fields other than $\mathbb{C}$
  - Return to self-product example
    - $\mathfrak{C}(E \times E)$ cannot be defined over the minimal field of definition of E
  - Elliptic curves and the ring class field
    - Give a condition on the fields of definition of two elliptic curves with class field theory
  - Finding other interesting examples
- Conclusions and future work

Field of definition

Issues extending to other fields

- Question: is there an analogue of higher-weight Jacobians for varieties of maximal Picard number over **fields other than** $\mathbb{C}$?

- For example, if we have an abelian surface $\mathscr{A}$ over a number field L, is there **an elliptic curve** $E_{\mathscr{A}}$ **over** L with some nice relationship to the Brauer group of $\mathscr{A}$?

- We would expect that $E_{\mathscr{A}} \otimes \mathbb{C} \cong \mathfrak{C}(\mathscr{A}_{\mathbb{C}})$, but sometimes **no such** $E_{\mathscr{A}}$ **over** L **exists!**

- This means though $\mathfrak{C}(\mathscr{A}_{\mathbb{C}})$ is algebraic (it is an abelian variety), the construction of $\mathfrak{C}(\mathscr{A}_{\mathbb{C}})$ is **not-so-algebraic**

Field of definition

j-invariants of elliptic curves

- For a complex elliptic curve E defined by an equation $y^2 = x^3 + ax + b$, the j-**invariant** is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

- A complex elliptic curve is determined up to isomorphism by its j-invariant, and E is always isomorphic over $\mathbb{C}$ to the elliptic curve defined by

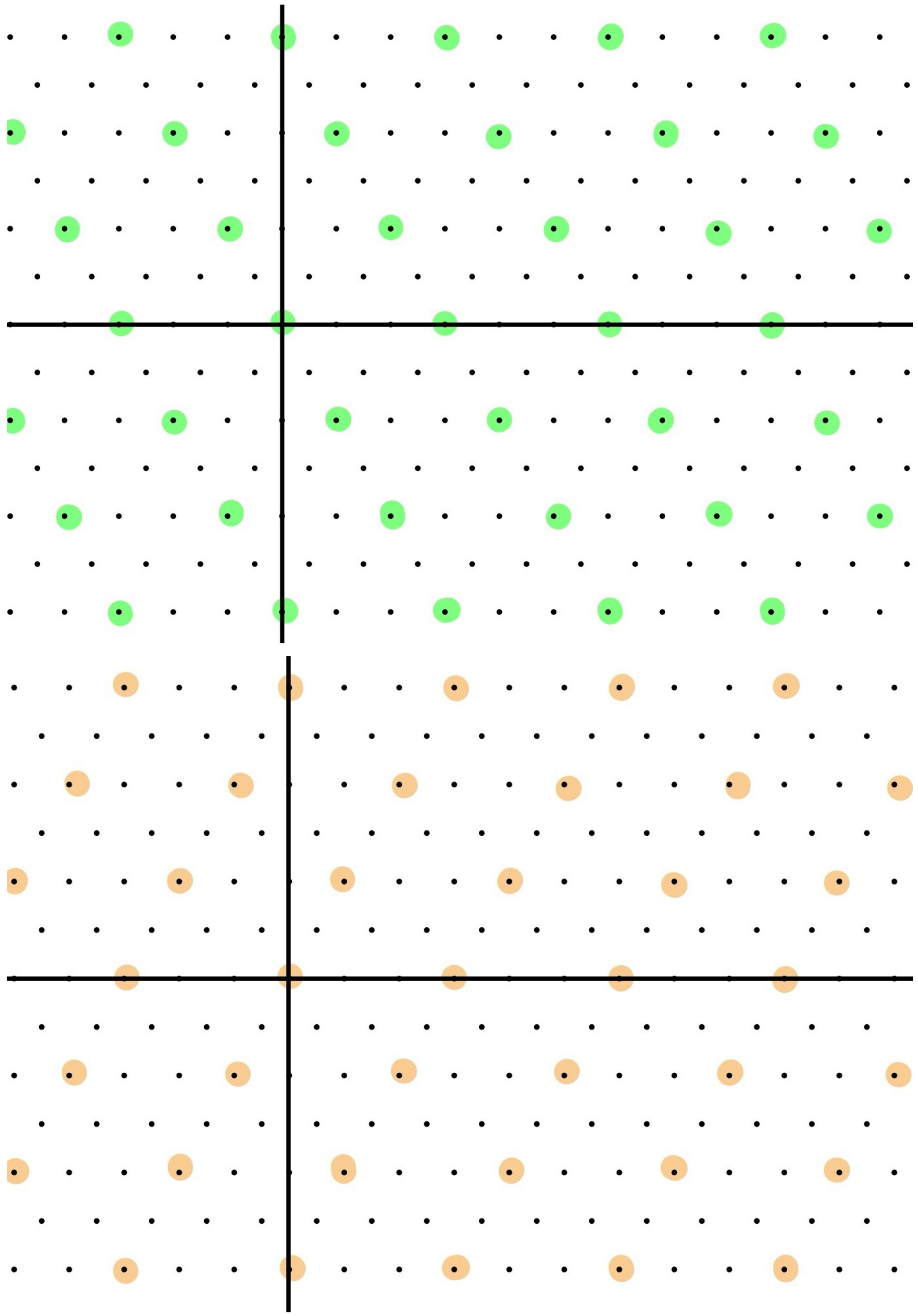
$$y^2 = x^3 + \frac{3j(E)}{1728 - j(E)}x + \frac{2j(E)}{1728 - j(E)}$$

- $\mathbb{Q}(j(E))$ is the "smallest" number field over which E can be defined

- We will give an example of an abelian surface $\mathscr{A}$ over a number field L so that $j(\mathfrak{C}(\mathscr{A}_{\mathbb{C}})) \notin L$

Field of definition

Return to self-product example



- Recall the elliptic curve $E = \mathbb{C}/\Lambda$ corresponding to the lattice $\Lambda = \langle 3, 2+\sqrt{-3}\rangle$
- $\mathfrak{C}(E \times E)$ is isomorphic to an elliptic curve $E' = \mathbb{C}/\Lambda'$ with $\Lambda' = \langle 3, 1+\sqrt{-3}\rangle$
- Λ, Λ' are **not homothetic** so $E \not\cong E'$
- By computation, $\mathbb{Q}(j(E)) = \mathbb{Q}(\zeta_3\sqrt[3]{2})$ and $\mathbb{Q}(j(E')) = \mathbb{Q}(\zeta_3^2\sqrt[3]{2})$
- Since $j(\mathfrak{C}(E \times E)) \notin \mathbb{Q}(j(E))$, we can define $E \times E$ over this field $\mathbb{Q}(j(E)) = \mathbb{Q}(\zeta_3\sqrt[3]{2})$, but we **can't define its 2-Jacobian!**

Field of definition

Self-product example and the class group

- We will find other examples of an elliptic curve E such that E can be defined over a number field L but $\mathfrak{C}(E \times E)$ cannot, using the class group of $\mathrm{End}(E)$

- Let's look at $\mathrm{Cl}(\mathrm{End}(E))$ for our previous example $E = \mathbb{C}/\Lambda$ where $\Lambda = \langle 3, 2 + \sqrt{-3} \rangle$

- In this case, $\mathrm{End}(E) = \mathbb{Z}[3\sqrt{-3}]$ and $\mathrm{Cl}(\mathbb{Z}[3\sqrt{-3}]) \cong \mathbb{Z}/3\mathbb{Z}$

- Note that $\mathbb{Z}[3\sqrt{-3}] = \langle 1, 3\sqrt{-3} \rangle$ is the lattice corresponding to the identity element and $[E], [E]^2 = [\mathfrak{C}(E \times E)] = [E']$ are the non-identity elements

- $\mathbb{Q}(j(\mathbb{C}/\langle 1, 3\sqrt{-3} \rangle)) = \mathbb{Q}(\sqrt[3]{2})$, which is distinct from both $\mathbb{Q}(j(E))$, $\mathbb{Q}(j(E'))$

- All elements of the class group have **different minimal fields of definition**

Field of definition

Elliptic curves and the ring class field

- For two CM elliptic curves E_1, E_2, we want to describe when $j(E_2) \in \mathbb{Q}(j(E_1))$
- For **any** elliptic curve E with CM by $R \subseteq R \otimes_{\mathbb{Z}} \mathbb{Q} = K$ the extension $K(j(E))$ is equal to the **ring class field** L_R
- L_R is a degree $n = |\mathrm{Cl}(R)|$ extension of K
- We use class field theory to prove a **field of definition condition**:

Proposition (Devadas-Lieblich). If E_1, E_2 are two elliptic curves with CM by R then either
(i) $[E_1]^2 = [E_2]^2 \in \mathrm{Cl}(R)$ and $\mathbb{Q}(j(E_1)) = \mathbb{Q}(j(E_2))$ is a degree n extension of $\mathbb{Q}$, or
(ii) $j(E_2) \notin \mathbb{Q}(j(E_1))$ and $\mathbb{Q}(j(E_1), j(E_2)) = L_R$

Field of definition

Other interesting examples

- We can find many cases of an elliptic curve E such that E can be defined over a number field L but $\mathfrak{C}(E \times E)$ cannot

- If E is a CM elliptic curve such that the order of $[E] \in \mathrm{Cl}(\mathrm{End}(E))$ is greater than 2, then

$$[\mathfrak{C}(E \times E)]^2 = [E]^4 \neq [E]^2$$

- This means that $j(\mathfrak{C}(E \times E)) \notin \mathbb{Q}(j(E)) = L$ by our field of definition condition

- It is still true that both $j(E)$ and $j(\mathfrak{C}(E \times E)) \in L_{\mathrm{End}(E)}$, the ring class field

Field of definition

Isomorphism classes of abelian surfaces

- To determine the "field of definition" of an abelian surface of maximal Picard rank $\mathscr{A} = E_1 \times E_2$ we must determine when $\mathscr{A} \cong \mathscr{A}'$ for another abelian surface of maximal Picard rank

Theorem (Shioda-Mitani). If $E_1, \ldots, E_4$ are pairwise isogenous CM elliptic curves, then $E_1 \times E_2 \cong E_3 \times E_4$ if and only if
(i) $\mathrm{End}(E_1) \cap \mathrm{End}(E_2) = \mathrm{End}(E_3) \cap \mathrm{End}(E_4)$
(ii) $\mathrm{End}(E_1) + \mathrm{End}(E_2) = \mathrm{End}(E_3) + \mathrm{End}(E_4)$
(iii) $\mathfrak{C}(E_1 \times E_2) \cong \mathfrak{C}(E_3 \times E_4)$

- So in particular, if $\mathrm{End}(E_1) = \mathrm{End}(E_2) = R$ then $E_1 \times E_2 \cong E_3 \times E_4$ if and only if both
  (1) $R = \mathrm{End}(E_3) = \mathrm{End}(E_4)$
  (2) $\mathfrak{C}(E_1 \times E_2) \cong \mathfrak{C}(E_3 \times E_4)$, or $[E_1][E_2] = [E_3][E_4] \in \mathrm{Cl}(R)$

Field of definition

Interesting examples, part 2

- If E is a CM elliptic curve such that the order of $[E] \in \mathrm{Cl}(\mathrm{End}(E))$ is greater than 2, then $j(\mathfrak{C}(E \times E)) \notin \mathbb{Q}(j(E))$
- However, by the **Shioda-Mitani condition**, $E \times E$ is isomorphic to various other products $E_1 \times E_2$ with

$$\mathrm{End}(E_1) = \mathrm{End}(E_2) = \mathrm{End}(E) \text{ and } [E]^2 = [E_1][E_2]$$

- If $[E] \in \mathrm{Cl}(\mathrm{End}(E))$ has order 4, then

$$E \times E \cong \mathbb{C}/\mathrm{End}(E) \times \mathfrak{C}(E \times E)$$

- By our field of definition condition,

$$\mathbb{Q}(j(\mathfrak{C}(E \times E))) = \mathbb{Q}(j(\mathbb{C}/\mathrm{End}(E)))$$

- This means $E \times E$ and $\mathfrak{C}(E \times E)$ can both be defined over a field smaller than the ring class field

Theorem (Devadas-Lieblich). If $\mathscr{A} = E \times E'$ is an abelian surface of maximal Picard rank such that such that $\mathrm{End}(E) = \mathrm{End}(E') = R$, then there exist elliptic curves E_1, E_2 with $\mathscr{A} \simeq E_1 \times E_2$ such that $\mathbb{Q}(j(E_1)) = \mathbb{Q}(j(E_2)) = \mathbb{Q}(j(\mathfrak{C}(\mathscr{A})))$ if and only if the group element $[\mathfrak{C}(\mathscr{A})] \in \mathrm{Cl}(R)$ has order ≤ 2.

Conclusion

In summary

- For $X \cong E_1 \times E_2 \times \ldots \times E_n$ pairwise isogenous CM elliptic curves with $\mathrm{End}(E_i) = R_i$, then $\mathfrak{C}_n(X)$ is an elliptic curve with CM by $R_1 + \ldots + R_n \subseteq K = \mathrm{End}(E_i) \otimes_{\mathbb{Z}} \mathbb{Q}$

- For $m < n$ we have

$$\mathfrak{C}_m(X) \cong \prod_{i_1 < i_2 < \ldots < i_m} \mathfrak{C}_m\left(E_{i_1} \times \ldots \times E_{i_m}\right)$$

  which is also a complex abelian variety of maximal Picard number

- This analytic construction produces objects which are "very algebraic"!

- We gave examples of some abelian surfaces $\mathscr{A}$ over a number field L so that $j(\mathfrak{C}(\mathscr{A}_{\mathbb{C}})) \notin L$

- This means though $\mathfrak{C}(\mathscr{A}_{\mathbb{C}})$ is algebraic (it is an abelian variety), the construction of $\mathfrak{C}(\mathscr{A}_{\mathbb{C}})$ is **not quite algebraic**

Conclusion

Future work

- By the Shioda-Mitani condition, an abelian surface of maximal Picard number $\mathscr{A} = E_1 \times E_2$ is described up to isomorphism by $\mathfrak{C}(\mathscr{A})$ and its **degree of primitivity**.
  - For primitive Picard-maximal abelian surfaces, we can fully describe the minimal field over which $\mathscr{A}, \mathfrak{C}(\mathscr{A})$ can both be defined. What about the non-primitive case?
  - Can we extend this field of definition condition further to Kummer surfaces and general singular K3 surfaces?
- Are there complex surfaces such that $\mathfrak{C}(S)$ is a complex torus but not an abelian variety?
- Beauville computed $\mathfrak{C}(C_1 \times C_2)$ up to isogeny for products of curves with maximal Picard number
  - Can we compute these $\mathfrak{C}(C_1 \times C_2)$ up to isomorphism? What about $\mathfrak{C}_m(C_1 \times \ldots \times C_n)$ for products of n curves?

Conclusion

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Conclusion

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