A Noetherian counterexample to Theorem B for Henselian schemes

Sheela Devadas

The blog post [Jon18] by Johan de Jong describes a non-Noetherian counterexample to Theorem B for Henselian schemes.

Example 1. Let (R, J) be the Henselization of the pair $(\mathbf{C}[x, y], (xy(x + y - 1)))$.

If we let A be the integral closure of R in the algebraic closure $K = \overline{\mathbf{C}(x, y)}$ and let $I = JA = xy(x + y - 1)A$, then the pair (A, I) is Henselian as the map $R \to A$ is integral.

The cohomology group $\mathrm{H}^1(\mathrm{Sph}(A, I), \mathscr{O}_{\mathrm{Sph}(A,I)})$ is shown to be nonzero in [Jon18]. For a more detailed proof that the cohomology is nonzero, see [Dev22, Proposition 3.1.15]. ■

Proposition 2. *There exists a Noetherian Henselian pair* (R_0, I_0) *such that the cohomology* $\mathrm{H}^1(\mathrm{Sph}(R_0, I_0), \mathscr{O}_{\mathrm{Sph}(R_0,I_0)})$ *is nonzero. Thus we have a Noetherian counterexample to Theorem B for Henselian schemes.*

Proof. Let R, J, A, I be as in Example 1. We can write $A = \varinjlim R_i$ as the filtered direct limit of its subalgebras R_i which are module-finite over R (and hence Noetherian).

If $I_i = JR_i = xy(x+y-1)R_i$, then each pair (R_i, I_i) is Henselian since the maps $R \to R_i$ are integral. Then $\mathrm{Sph}(A, I) = \varprojlim \mathrm{Sph}(R_i, I_i)$, so we can apply [Dev23, Lemma A.2] to "pull the limit into the cohomology" and deduce

$$\varinjlim \mathrm{H}^1(\mathrm{Sph}(R_i, I_i), \mathscr{O}_{\mathrm{Sph}(R_i,I_i)}) = \mathrm{H}^1(\mathrm{Sph}(A, I), \mathscr{O}_{\mathrm{Sph}(A,I)}) \neq 0.$$

Thus for some i_0 we have $\mathrm{H}^1(\mathrm{Sph}(R_{i_0}, I_{i_0}), \mathscr{O}_{\mathrm{Sph}(R_{i_0},I_{i_0})}) \neq 0$. □

References

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